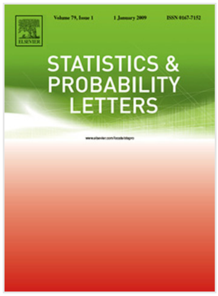
Accepted Manuscript



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PII: S0167-7152(16)30044-X
DOI: http://dx.doi.org/10.1016/j.spl.2016.04.023
Reference: STAPRO 7604

To appear in: *Statistics and Probability Letters*

Received date: 23 December 2015
Revised date: 27 April 2016
Accepted date: 27 April 2016

Consistency of the plug-in functional predictor of the Ornstein-Uhlenbeck process in Hilbert and Banach spaces

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Abstract: New results on functional prediction of the Ornstein-Uhlenbeck process in an autoregressive Hilbert-valued and Banach-valued frameworks are derived. Specifically, consistency of the maximum likelihood estimator of the autocorrelation operator, and of the associated plug-in predictor is obtained in both frameworks.

MSC 2010 subject classifications: Primary 60G10, 60G15; secondary 60F99, 60J05, 65F15.
Keywords and phrases: Autoregressive Hilbertian processes, Banach-valued autoregressive processes, Consistency, Maximum likelihood parameter estimator, Ornstein-Uhlenbeck process.

1. Introduction

This paper derives new results in the context of linear processes in function spaces. An extensive literature has been developed in this context, in the last few decades (see, for example, Bosq (2000); Ferraty and Vieu (2006); Ramsay and Silverman (2005), among others). In particular, the problem of functional prediction of linear processes in Hilbert and Banach spaces has been widely addressed. We refer to the reader to the papers by: Bensmain and Mourid (2003); Bosq (1996); Bosq (2002); Bosq (2004); Bosq (2007); Dedecker and Merlevede (2003); Dehling and Sharipov (2005); Glendinning and Fleet (2007); Guillas (2000); Guillas (2001); Kargin and Onatski (2008); Labbas and Mourid (2003); Marion and Pumo (2004); Mas (2002); Mas (2004); Mas (2007); Mas and Menneteau (2003); Mas and Pumo (2007); Menneteau (2005); Mourid (2002); Mourid (2004); Mokhtari and Mourid (2002); Pumo (1998); Rachedi (2004); Rachedi (2005); Rachedi and Mourid (2003); Ruiz-Medina (2012); Turbillon, Marion and Pumo (2007); Turbillon et. al (2008), and the references therein. In the above-mentioned papers, different projection methodologies have been adopted in the derivation of the main asymptotic properties of the formulated functional parameter estimators and predictors. Particularly, Bosq (2000) and Bosq and Blanke (2007) apply Functional Principal Component Analysis; Antoniadis, Paparoditis and Sapatinas (2006) and Antoniadis and Sapatinas (2003) consider wavelet bases; Laukaitis, Vasilecas and Laukaitis (2009) propose wavelet estimation methods. Applications of these functional estimation results can be found in the papers by: Antoniadis and Sapatinas (2003); Damon and Guillas (2002); Hormann and Kokoszka (2011); Laukaitis (2008); Ruiz-Medina and Salmerón (2009), among others.

We pay attention here to the problem of functional prediction of the Ornstein-Uhlenbeck (O.U.) process (see, for example, Uhlenbeck and Ornstein (1930), and Wang and Uhlenbeck (1945), for its introduction and properties). See also Doob (1942) for the classical definition of O.U. process from the Langevin (linear) stochastic differential equation. We can find in Kutoyants (2004) and Liptser and Shiraev (2001) an explicit expression of the maximum likelihood estimator (MLE) of the scale parameter θ, characterizing its covariance function. Its strong consistency is proved, for instance, in Kleptsyna and Le Breton (2002). We formulate here the O.U. process as an Autoregressive Hilbertian process of order one (ARH(1) process), and as an Autoregressive Banach -valued process of order one (ARB(1) process). Consistency of the MLE of θ is applied to prove consistency of the corresponding MLE of the autocorrelation operator of the O.U. process. We adopt the methodology applied in Bosq (1991), since our interest relies on forecasting the values of the O.U. process over an entire time interval. Specifically, considering the O.U. process $\{\xi_t\}_{t\in\mathbb{R}}$, on the basic probability space $(\Omega, \mathcal{A}, P)$, we can define

$$X_n(t) = \xi_{nh+t}, \quad 0 \leq t \leq h, \; n \in \mathbb{Z}, \tag{1}$$

satisfying

$$X_n(t) \quad = \quad \xi_{nh+t} = \int_{-\infty}^{nh+t} e^{-\theta(nh+t-s)} dW_s = \rho_\theta\left(X_{n-1}\right)(t) + \varepsilon_n(t), \quad n \in \mathbb{Z}, \tag{2}$$

with

$$\rho_\theta(x)(t) = e^{-\theta t} x(h), \quad \rho_\theta\left(X_{n-1}\right)(t) = e^{-\theta t}\int_{-\infty}^{nh} e^{-\theta(nh-s)} dW_s, \quad \varepsilon_n(t) = \int_{nh}^{nh+t} e^{-\theta(nh+t-s)} dW_s, \tag{3}$$

for $0 \leq t \leq h$. Thus, $X = (X_n, \; n \in \mathbb{Z})$ satisfies the ARH(1) equation (2) (see also equation (5) below for its general definition). The real separable Hilbert space H is given by $H = L^2\left([0,h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$, where $\beta_{[0,h]}$ is the Borel σ-algebra generated by the subintervals in $[0,h]$, λ is the Lebesgue measure, and $\delta_{(h)}(s) = \delta(s-h)$ is the Dirac measure at point h. The associated norm

$$\|f\|_{H=L^2\left([0,h],\beta_{[0,h]},\lambda+\delta_{(h)}\right)} = \sqrt{\int_0^h f^2(t)dt + f^2(h)}$$

establishes the equivalent classes of functions given by the relationship $f \sim_{\lambda+\delta_{(h)}} g$ if and only if $\left(\lambda + \delta_{(h)}\right)(\{t : f(t) \neq g(t)\}) = 0$, with

$$\left(\lambda + \delta_{(h)}\right)(\{t : f(t) \neq g(t)\}) = 0 \Leftrightarrow \lambda(\{t : f(t) \neq g(t)\}) = 0 \text{ and } f(h) = g(h), \tag{4}$$

where, as before, $\delta_{(h)}$ is the Dirac measure. We will prove, in Lemma 1 below, that $X = (X_n, \; n \in \mathbb{Z})$, constructed in (1) from the O.U. process, satisfying equations (2)–(3), is the unique stationary solution to equation (2), in the space $H = L^2\left([0,h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$, admitting a MAH($\infty$) representation. Similarly, in Lemma 4 below, we will prove that $X = (X_n, \; n \in \mathbb{Z})$, constructed in (1) from the O.U. process, satisfying equations (2)–(3), is the unique stationary solution to equation (2), admitting a MAB(∞) representation, in the space $B = \mathcal{C}([0,h])$, the Banach space of continuous functions, whose support is the interval $[0,h]$, with the supremum norm.

The main results of this paper provide the almost surely convergence to ρ_θ of the MLE $\rho_{\widehat{\theta}}$ of ρ_θ, in the norm of $\mathcal{L}(H)$, the space of bounded linear operators in the Hilbert space H (respectively, in the norm of $\mathcal{L}(B)$, the space of bounded linear operators in the Banach space B). The convergence in probability of the associated plug-in ARH(1) and ARB(1) predictors, i.e., the convergence in probability of $\rho_{\widehat{\theta}}(X_{n-1})$ to $\rho_\theta(X_{n-1})$ in H and B, respectively, is proved as well.

The outline of this paper is as follows. In Section 2, the main results of this paper are obtained. Specifically, Section 2.1 provides the definition of O.U. process as an ARH(1) process. Strong consistency in $\mathcal{L}(H)$ of the estimator of the autocorrelation operator is derived in Section 2.2. Consistency in H of the associated plug-in ARH(1) predictor is then established in Section 2.3. The corresponding results in Banach spaces are given in Section 2.4. For illustration purposes, a simulation study is undertaken in Section 3. Final comments can be found in Section 4. (The basic preliminary elements applied in the proof of the main results of this paper and the proof of Lemma 1 can be found in the supplementary material).

2. Prediction of O.U. process in Hilbert and Banach spaces

In this section, we consider H to be a real separable Hilbert space. Recall that a zero-mean ARH(1) process $X = (X_n,\ n \in \mathbb{Z})$, on the basic probability space $(\Omega, \mathcal{A}, P)$, satisfies (see Bosq (2000))

$$X_n = \rho\left(X_{n-1}\right) + \varepsilon_n, \quad n \in \mathbb{Z}, \tag{5}$$

where ρ denotes the autocorrelation operator of process X. Here, $\varepsilon = (\varepsilon_n,\ n \in \mathbb{Z})$ is assumed to be a strong-white noise, i.e., ε is a Hilbert-valued zero-mean stationary process, with independent and identically distributed components in time, and with $\sigma^2 = E\|\varepsilon_n\|_H^2 < \infty$, for all $n \in \mathbb{Z}$.

2.1. O.U. process as ARH(1) process

As commented in the Introduction, equations (1)–(3) provide the definition of O.U. process as an ARH(1) process, with $H = L^2\left([0,h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$. The norm in the space $H = L^2\left([0,h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$ of $\rho_\theta(x)$, with ρ_θ introduced in (3) and $x \in H$, is given by

$$\|\rho_\theta(x)\|_H^2 \ = \ \int_0^h |\rho_\theta\left(x\right)\left(t\right)|^2 d\left(\lambda + \delta_{(h)}\right)(t) = \int_0^h |\rho_\theta\left(x\right)\left(t\right)|^2 dt + |\rho_\theta\left(x\right)\left(h\right)|^2, \tag{6}$$

for each h strictly positive. The following lemma provides, for each $k \geq 1$, the exact value of the norm of ρ_θ^k, in the space of bounded linear operators on $L^2\left([0,h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$. As a direct consequence, the existence of k_0 such that $\|\rho_\theta^k\|_{\mathcal{L}(H)} < 1$, for $k \geq k_0$, is also derived for $\theta > 0$.

Lemma 1 *Let us consider, for $n \in \mathbb{Z}$, X_n satisfying equations (1)–(3). For each $k \geq 1$, the norm of ρ_θ^k is given by*

$$\|\rho_\theta^k\|_{\mathcal{L}(H)} = \sqrt{e^{-2\theta(k-1)h}\frac{1 + e^{-2\theta h}\left(2\theta - 1\right)}{2\theta}} = e^{-\theta(k-1)h}\|\rho_\theta\|_{\mathcal{L}(H)}. \tag{7}$$

Furthermore, for $k \geq k_0 = \left[\frac{1}{\theta} + 1\right]^+$,

$$\|\rho_\theta^k\|_{\mathcal{L}(H)} < 1, \tag{8}$$

where $[t]^+$ denotes the closest upper integer of t, for every $t \in \mathbb{R}_+$.

The proof of this lemma can be found in the supplementary material.

Remark 1 *From equation (8), applying Theorem 3.1. in Bosq (2000), p. 74, Lemma 1 implies that $X = (X_n,\ n \in \mathbb{Z})$, constructed in (1) from O.U. process, defines the unique stationary solution to equation (2) in the space $H = L^2\left([0,h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$, admitting the MAH($\infty$) representation*

$$X_n = \sum_{k=0}^{+\infty} \rho_\theta^k\left(\varepsilon_{n-k}\right), \quad n \in \mathbb{Z},\ \rho_\theta \in \mathcal{L}\left(H\right). \tag{9}$$

Remark 2 *Note that, for all $x \in H$, and $k \geq 2$, $\|\rho_\theta^k\|_{\mathcal{L}(H)} \leq \left[\|\rho_\theta\|_{\mathcal{L}(H)}\right]^k$.*

2.2. Functional parameter estimation and consistency

We now prove strong consistency of the estimator $\rho_{\widehat{\theta}_n}$ of operator ρ_θ in $\mathcal{L}(H)$, with, as before, $H = L^2\left([0,h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$, and $\widehat{\theta}_n$ denoting the MLE of θ, based on the observation of O.U. process on the interval $[0,T]$, with $T = nh$. Note that, from equation (3), for all $x \in H$, and for a given sample size n, $\rho_{\widehat{\theta}_n}(x) = e^{-\widehat{\theta}_n t} x(h)$, where the MLE of θ is given, for $T = nh$, by

$$\widehat{\theta}_T = \frac{1 + \frac{\xi_0^2}{T} - \frac{\xi_T^2}{T}}{\frac{2}{T}\int_0^T \xi_t^2 dt}, \quad T > 0, \tag{10}$$

with ξ_t, $t \in [0,T]$, being the observed values of the O.U. process over the interval $[0,T]$. Thus, $\rho_{\widehat{\theta}_n}$ is introduced in an abstract way, since it can only be explicitly computed, for each particular function $x \in H$ considered. However, the norm $\|\rho_\theta - \rho_{\widehat{\theta}_n}\|_{\mathcal{L}(H)}$ is explicitly computed in equation (13) below.

The following results will be applied in the proof of Proposition 1.

Lemma 2 *If $t \in [0,+\infty)$, it holds that $|e^{-ut} - e^{-vt}| \leq |u - v|t$, for any $u, v \geq 0$.*

The proof of this lemma is given in the supplementary material.

Theorem 1 *(see Kleptsyna and Le Breton (2002), Proposition 2.2, p. 4, and Kutoyants (2004), p. 63 and p. 117). The MLE of θ defined in equation (10) is strongly consistent, i.e.,*

$$\lim_{T\to\infty} \widehat{\theta}_T = \theta \quad \text{almost surely.} \tag{11}$$

The proof follows from Ibragimov-Khasminskii's Theorem.

Proposition 1 *Let H be the space $L^2\left([0,h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$. Then, the estimator $\rho_{\widehat{\theta}_n}$ of operator ρ_θ, based on the MLE $\widehat{\theta}_n$ of θ, is strongly consistent in $\mathcal{L}(H)$, i.e.,*

$$\|\rho_\theta - \rho_{\widehat{\theta}_n}\|_{\mathcal{L}(H)} \to^{a.s.} 0. \tag{12}$$

Proof. The following straightforward almost surely identities are obtained:

$$\begin{aligned}
\|\rho_\theta - \rho_{\widehat{\theta}_n}\|_{\mathcal{L}(H)} &= \sup_{x\in H}\left\{\frac{\|\left(\rho_\theta - \rho_{\widehat{\theta}_n}\right)(x)\|_H}{\|x\|_H}\right\} \\
&= \sup_{x\in H}\left\{\sqrt{\frac{\int_0^h |\left(\rho_\theta - \rho_{\widehat{\theta}_n}\right)(x)(t)|^2 d\left(\lambda + \delta_{(h)}\right)(t)}{\int_0^h |x(t)|^2 d\left(\lambda + \delta_{(h)}\right)(t)}}\right\} \\
&= \sup_{x\in H}\left\{\sqrt{x^2(h)\frac{\int_0^h \left(e^{-\theta t} - e^{-\widehat{\theta}_n t}\right)^2 dt + \left(e^{-\theta h} - e^{-\widehat{\theta}_n h}\right)^2}{\int_0^h x^2(t)\,dt + x^2(h)}}\right\}
\end{aligned}$$

$$= \sqrt{\int_0^h \left(e^{-\theta t} - e^{-\widehat{\theta}_n t}\right)^2 dt + \left(e^{-\theta h} - e^{-\widehat{\theta}_n h}\right)^2}, \tag{13}$$

where the last identity is obtained in a similar way to equation (7) in Lemma 1 (see also equations (26)–(30) in the supplementary material).

From Lemma 2 and equation (13), for n sufficiently large, we have

$$\begin{aligned} \|\rho_\theta - \rho_{\widehat{\theta}_n}\|_{\mathcal{L}(H)} &\leq \sqrt{\int_0^h t^2|\theta - \widehat{\theta}_n|^2 dt + h^2|\theta - \widehat{\theta}_n|^2} = |\theta - \widehat{\theta}_n|\sqrt{\int_0^h t^2 dt + h^2} \\ &= |\theta - \widehat{\theta}_n| h \sqrt{\frac{h}{3} + 1} \quad \text{almost surely.} \end{aligned} \tag{14}$$

The strong-consistency of $\rho_{\widehat{\theta}_n}$ in $\mathcal{L}(H)$ directly follows from Theorem 1 and equation (14). ■

Remark 3 *From Proposition 2.3(i) in Kleptsyna and Le Breton (2002), p. 5 (see also Theorem 2 below), the MLE $\widehat{\theta}_T$ of θ satisfies*

$$E\left[\left(\theta - \widehat{\theta}_T\right)^2\right] = \mathcal{O}\left(\frac{2\theta}{T}\right), \quad T \to \infty. \tag{15}$$

In addition, from equation (14), considering $T = nh$, $h > 0$,

$$E\left[\|\rho_\theta - \rho_{\widehat{\theta}_n}\|^2_{\mathcal{L}(H)}\right] \leq E\left[|\theta - \widehat{\theta}_n|^2\right] h^2 \left(\frac{h}{3} + 1\right). \tag{16}$$

Equations (15) and (16) lead to

$$E\left[\|\rho_\theta - \rho_{\widehat{\theta}_n}\|^2_{\mathcal{L}(H)}\right] \leq G(\theta, \widehat{\theta}_n, h),$$

with $G(\theta, \widehat{\theta}_n, h) = \mathcal{O}\left(\frac{2\theta}{n}\right)$, $n \to \infty$. Therefore, the functional parameter estimator $\rho_{\widehat{\theta}_n}$ of ρ_θ is $\sqrt{n}$-consistent.

2.3. Consistency of the plug-in ARH(1) predictor

Let us consider the plug-in ARH(1) predictor $\widehat{X}_n$, constructed from the MLE $\rho_{\widehat{\theta}_n}$ of ρ_θ in Proposition 1, given by

$$\widehat{X}_n(t) = \rho_{\widehat{\theta}_n}(X_{n-1})(t) = e^{-\widehat{\theta}_n t} X_{n-1}(h), \quad 0 \leq t \leq h, \ n \in \mathbb{Z}. \tag{17}$$

Corollary 1 below provides the consistency of $\widehat{X}_n$, given in equation (17), from Proposition 1 by applying the following lemma and theorem.

Lemma 3 *Let $\{Z_n\}_{n\in\mathbb{Z}}$ be a sequence of random variables such that $Z_n \sim \mathcal{N}\left(0, \frac{1}{2\theta}\right)$, with $\theta > 0$, and let $\{Y_n\}_{n\in\mathbb{Z}}$ be another sequence of random variables such that $\sqrt{\ln(n)}Y_n \to^p 0$, $n \to \infty$. Then, $Y_n|Z_n| \to^p 0$, $n \to \infty$, where, as usual, $\to^p$ indicates convergence in probability.*

The proof of this lemma can be found in the supplementary material.

Theorem 2 *Let $\widehat{\theta}_T$ be the MLE of θ defined in equation (10), with $\theta > 0$. Hence,*

$$E\left[\left(\theta - \widehat{\theta}_T\right)^2\right] = \mathcal{O}\left(\frac{2\theta}{T}\right), \quad T \to \infty. \tag{18}$$

In particular,

$$\lim_{T\to\infty} E\left[\left(\theta - \widehat{\theta}_T\right)^2\right] = 0. \tag{19}$$

The proof of this result is given in in Proposition 2.3(i) in Kleptsyna and Le Breton (2002), p. 5.

Corollary 1 *Let $H = L^2\left([0,h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$ be the Hilbert space introduced above. Then, the plug-in ARH(1) predictor (17) of O.U. process is consistent in H, i.e.,*

$$\|\left(\rho_\theta - \rho_{\widehat{\theta}_n}\right)(X_{n-1})\|_H \to^p 0. \tag{20}$$

Proof. By definition,

$$\|\left(\rho_\theta - \rho_{\widehat{\theta}_n}\right)(X_{n-1})\|_H = |X_{n-1}(h)|\sqrt{\int_0^h \left(e^{-\theta t} - e^{-\widehat{\theta}_n t}\right)^2 dt + \left(e^{-\theta h} - e^{-\widehat{\theta}_n h}\right)^2}. \tag{21}$$

From equations (13)–(14) and (21), we then obtain, for n sufficiently large,

$$\|\left(\rho_\theta - \rho_{\widehat{\theta}_n}\right)(X_{n-1})\|_H \leq |X_{n-1}(h)||\theta - \widehat{\theta}_n|h\sqrt{\frac{h}{3}+1} \quad \text{a.s.} \tag{22}$$

Let us set $\{Y_n\}_{n\in\mathbb{Z}} = \left\{|\theta - \widehat{\theta}_n|h\sqrt{\frac{h}{3}+1}\right\}_{n\in\mathbb{Z}}$ and $\{Z_n\}_{n\in\mathbb{Z}} = \{X_{n-1}(h)\}_{n\in\mathbb{Z}}$, with $X_{n-1} \sim \mathcal{N}\left(0, \frac{1}{2\theta}\right)$, for every $n \in \mathbb{Z}$. From Theorem 1, $Y_n \to^{a.s.} 0$, $n \to \infty$. Hence, to apply Lemma 3, we need to prove that

$$\sqrt{ln\,(n)}Y_n \to^p 0, \quad n \to \infty. \tag{23}$$

From Chebyshev's inequality and Theorem 2, we get, for all $\varepsilon > 0$,

$$\mathcal{P}\left(|\theta - \widehat{\theta}_n|\sqrt{ln\,(n)}h\sqrt{\frac{h}{3}+1} \geq \varepsilon\right) \leq \frac{h^2\left(\frac{h}{3}+1\right)ln\,(n)\,E\left[|\theta - \widehat{\theta}_n|^2\right]}{\varepsilon^2} \to^{n\to+\infty} 0. \tag{24}$$

Therefore, from Lemma 3, we obtain the convergence in probability of $\|\left(\rho_\theta - \rho_{\widehat{\theta}_n}\right)(X_{n-1})\|_H$ to zero.

■

2.4. Prediction of O.U. process in $B = \mathcal{C}([0,h])$

As before, let B be now the Banach space of continuous functions, whose support is the interval $[0,h]$, with the supremum norm, denoted as $\mathcal{C}([0,h])$. The following lemma states that $\|\rho_\theta^k\|_{\mathcal{L}(B)} \leq 1$, for $\theta > 0$, and for every $k \geq 1$, with $\mathcal{L}(B)$ being the space of bounded linear operators on the Banach space $B = \mathcal{C}([0,h])$, and ρ_θ being introduced in equation (3). Consequently, considering condition (c_1) in Bosq (2000), p. 74, with $a = 2, b = 1/2$ and $j = 1$, from Lemma 3.1 and Theorem 3.1, in Bosq (2000), pp. 74–75, $X = (X_n,\ n \in \mathbb{Z})$, constructed in (1) from O.U. process, defines the unique stationary solution to equation (2), in the Banach space $B = \mathcal{C}([0,h])$, admitting a MAB(∞) representation.

Lemma 4 *Let ρ_θ introduced in (3), defined on $B = \mathcal{C}([0,h])$. Then, for $k \geq 1$, $\|\rho_\theta^k\|_{\mathcal{L}(B)} \leq 1$, with $\theta > 0$.*

Proof.
From

$$\rho_\theta^k(x)(t) = e^{-\theta t} e^{-\theta(k-1)h} x(h),$$

for each $k \geq 1$ and $\theta > 0$, we have

$$\begin{aligned}
\|\rho_\theta^k\|_{\mathcal{L}(B)} &= \sup_{x \in B} \left\{ \frac{\|\rho_\theta^k(x)\|_B}{\|x\|_B} \right\} = \sup_{x \in B} \frac{\sup_{0 \leq t \leq h} |\exp(-\theta t)\exp(-\theta(k-1)h)x(h)|}{\sup_{0 \leq t \leq h} |x(t)|} \\
&= \sup_{x \in B} \frac{|x(h)| \exp(-\theta(k-1)h) \sup_{0 \leq t \leq h} \exp(-\theta t)}{\sup_{0 \leq t \leq h} |x(t)|} \leq \sup_{x \in B} \frac{|x(h)| \sup_{0 \leq t \leq h} \exp(-\theta t)}{|x(h)|} \\
&= \sup_{0 \leq t \leq h} \exp(-\theta t) = 1. \qquad (25)
\end{aligned}$$

■

We now check strong consistency of the MLE $\rho_{\widehat{\theta}_n}$ of ρ_θ in $\mathcal{L}(B)$. From (25),

$$\|\rho_\theta - \rho_{\widehat{\theta}_n}\|_{\mathcal{L}(B)} \leq \sup_{0 \leq t \leq h} |e^{-\theta t} - e^{-\widehat{\theta}_n t}|, \quad \text{a.s.} \qquad (26)$$

From Lemma 2, for n sufficiently large, we then have

$$\|\rho_\theta - \rho_{\widehat{\theta}_n}\|_{\mathcal{L}(B)} \leq h|\theta - \widehat{\theta}_n|, \quad \text{a.s.} \qquad (27)$$

Theorem 1 then leads to the desired result on strong consistency of the estimator $\rho_{\widehat{\theta}_n}$ of ρ_θ in $\mathcal{L}(B)$. Furthermore, from Theorem 2 , in a similar way to Remark 3, $\sqrt{n}$-consistency of $\rho_{\widehat{\theta}_n}$ in $\mathcal{L}(B)$ also follows from equations (18) and (27).

Similarly to Corollary 1, in the following result, the consistency, in the Banach space $B = C([0,h])$, of the plug-in predictor (17) is obtained.

Corollary 2 *The ARB(1) plug-in predictor (17) of a zero-mean O.U. process is consistent in $B = C([0,h])$, i.e., as $n \to \infty$,*

$$\| \left(\rho_\theta - \rho_{\widehat{\theta}_n}\right)(X_{n-1}) \|_B \to^p 0. \qquad (28)$$

Proof. From Lemma 2, for n sufficiently large, and for each $h > 0$,

$$\| \left(\rho_\theta - \rho_{\widehat{\theta}_n}\right)(X_{n-1}) \|_B = \sup_{0 \leq t \leq h} |e^{-\theta t} - e^{-\widehat{\theta}_n t}||X_{n-1}(h)| \leq h|\theta - \widehat{\theta}_n||X_{n-1}(h)|, \quad \text{a.s.} \qquad (29)$$

As derived in the proof of Corollary 1, from Theorem 2, the random sequence $\{Y_n\}_{n \in \mathbb{Z}} = \left\{h|\theta - \widehat{\theta}_n|\right\}_{n \in \mathbb{Z}}$ is such that $\sqrt{ln(n)} Y_n \leq \sqrt{\frac{h}{3} + 1}\sqrt{ln(n)} Y_n \to^p 0$, $n \to \infty$. Moreover, $\{Z_n\}_{n \in \mathbb{Z}} = \{X_{n-1}(h)\}_{n \in \mathbb{Z}}$ is such that $X_{n-1}(h) \sim \mathcal{N}\left(0, \frac{1}{2\theta}\right)$. Lemma 3 then leads, as $n \to \infty$, to the desired convergence result from equation (29)

$$\| \left(\rho_\theta - \rho_{\widehat{\theta}_n}\right)(X_{n-1}) \|_B \leq Y_n |Z_n| \to^p 0. \qquad (30)$$

■

3. Simulations

In this section, a simulation study is undertaken to illustrate the asymptotic results presented in this paper about the MLE $\widehat{\theta}_n$ of θ, and the consistency of the ML functional parameter estimators of the autocorrelation operator, and the associated plug-in predictors, in the ARH(1) and ARB(1) frameworks.

3.1. Estimation of scale parameter θ

For simulation of the sample-paths of O.U. process, an extension of the Euler method, the Euler-Murayama method (see Kloeden and Platen (1992)) is applied, from the Langevin stochastic differential equation satisfied by the O.U. process $\{\xi_t,\ t \in [0,T]\}$

$$d\xi_t = -\theta\xi_t + dW_t, \quad \theta > 0, \quad t \in [0,T], \quad \xi_0 = x_0. \tag{31}$$

Thus, let $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ be a partition of real interval $[0,T]$. Then, (31) can be discretized as

$$\widehat{\xi}_{i+1} = \widehat{\xi}_i - \theta\widehat{\xi}_i + \Delta W_i, \quad \widehat{\xi}_0 = \xi_0 = 0, \tag{32}$$

where $\{\Delta W_i\}_{i=0,\dots,n-1}$ are i.i.d. Wiener increments, i.e., $\Delta W_i \sim \mathcal{N}(0,\Delta t) = \sqrt{\Delta t}\mathcal{N}(0,1)$. In the following, we take $\Delta t = 0.02$ as discretization step size, considering $N = 1000$ simulations of the O.U. process. In particular, Figure 1 shows some realizations of the discrete version of the solution to (31) generated from (32).

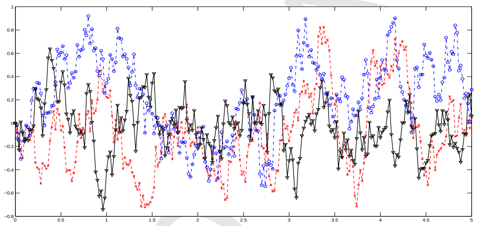

Figure 1: Three sample paths of O.U. process $\{\xi_t\}_{0\leq t\leq T}$ generated with $T = 5$, $\Delta t = 0.02$, $\theta = 5$, $\widehat{\xi}_0 = 0$

Let us first illustrate the asymptotic normal distribution of $\widehat{\theta}_T$, i.e., for T sufficiently large, we can consider $\widehat{\theta}_T \sim \mathcal{N}\left(\theta, \frac{2\theta}{T}\right)$ (see Theorem 1 in the supplementary material).

From equation (10), we take $\widehat{\theta}_T = \dfrac{-\displaystyle\int_0^T \xi_t d\xi_t}{\displaystyle\int_0^T \xi_t^2 dt}$ (see also equation (9) in the supplementary material), to compute the following approximation of the MLE $\widehat{\theta}_T$ of θ, for each one of the $N = 1000$ simulations performed of the O.U. process on $[0,T]$, and for each one of the six values of parameter θ considered:

$$\widehat{\theta}_T \simeq \frac{-\displaystyle\sum_{i=0}^{n-1} \widehat{\xi}_{t_i,s}(\theta)\left(\widehat{\xi}_{t_{i+1},s}(\theta) - \widehat{\xi}_{t_i,s}(\theta)\right)}{\displaystyle\sum_{i=0}^{n-1} \widehat{\xi}^2_{t_i,s}(\theta)\Delta t}, \; t_0 = 0,\ t_n = T,\ \Delta t = 0.02,\ \ s = 1,\dots,N, \tag{33}$$

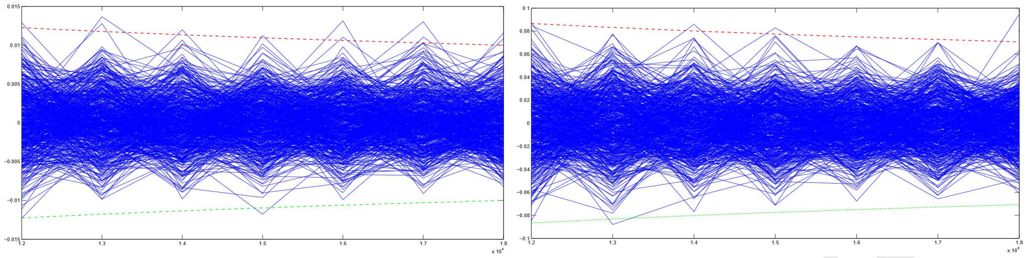

Figure 2: The values of $\widehat{\theta}_T - \theta$, based on $N = 1000$ simulations of the O.U. process over the interval $[0, T]$, for $T = 12000 + (l-1)1000$, $l = 1, \dots, 7$, are represented against the confidence bands given by $+3\sigma = 3\sqrt{\frac{2\theta}{T}}$ (upper dotted line) and $-3\sigma = -3\sqrt{\frac{2\theta}{T}}$ (lower dotted line), for $\theta = 0.1$ (at the left-hand side) and $\theta = 5$ (at the right-hand side)

where $\widehat{\xi}_{t_i,s}(\theta)$ represents the s-th discrete generation of the O.U. process, evaluated at time t_i, with covariance scale parameter θ, for $\theta = 0.1, 0.4, 0.7, 1, 2, 5$. Table 1 displays the empirical probabilities of the error $\widehat{\theta}_T - \theta$ to be within the band $\pm 3\sqrt{\frac{2\theta}{T}}$, from $N = 1000$ discrete simulations of the O.U. process, considering different sample sizes $T = 12000 + 1000(l-1)$, $l = 1, \dots, 7$, and the values $\theta = 0.1, 0.4, 0.7, 1, 2, 5$. Figure 2 displays the cases $\theta = 0.1$ (at the left-hand side) and $\theta = 5$ (at the right-hand side). It can be observed that, for each one of the sample sizes considered, $T = 12000 + 1000(l-1)$, $l = 1, \dots, 7$, approximately a 99% of the realizations of $\widehat{\theta}_T - \theta$ lie within the band $\pm 3\sqrt{\frac{2\theta}{T}}$, which supports the asymptotic Gaussian distribution.

$T\backslash\theta$	0.1	0.4	0.7	1	2	5
12000	0.9983	1	0.9983	0.9983	1	0.9983
13000	0.9967	0.9983	0.9983	1	0.9950	1
14000	0.9983	0.9967	1	0.9967	1	0.9983
15000	0.9983	0.9967	0.9983	0.9983	1	0.9983
16000	0.9967	0.9950	0.9967	0.9983	1	1
17000	0.9933	0.9983	1	0.9967	0.9950	1
18000	0.9967	0.9967	0.9950	1	1	0.9983

Table 1: Empirical probabilities of the error of the MLE of θ to lie within the band $\pm 3\sigma = \pm 3\sqrt{\frac{2\theta}{T}}$, for different sample sizes T, and values of parameter θ

Regarding asymptotic efficiency, stated in Theorem 2, from $N = 1000$ simulations of the O.U. process over the interval $[0, T]$, for $T = 50 + 250(l-1)$, $l = 1, \dots, 25$, the corresponding empirical mean square errors $EMSE_{N,T}(\theta) = \frac{1}{N}\sum_{s=1}^{N}\left(\theta - \widehat{\theta}_T(\omega_s)\right)^2$, $N = 1000$, $T = 50 + 250(l-1)$, $l = 1, \dots, 25$ (abbreviated as EMSE), considering the cases $\theta = 0.1, 0.4, 0.7, 1$, are displayed in Figure 3. Here, $\widehat{\theta}_T(\omega_s)$, $\omega_s \in \Omega$, $s = 1, \dots, N$, represent the respective approximated values (33) of the MLE of θ, computed from $\xi_{t_i,s}$, $s = 1, \dots, N$, $t_i \in [0, T]$, $i = 1, \dots, n$. It can be observed, from the results displayed in Figure 3, that Theorem 2 holds for T sufficiently large.

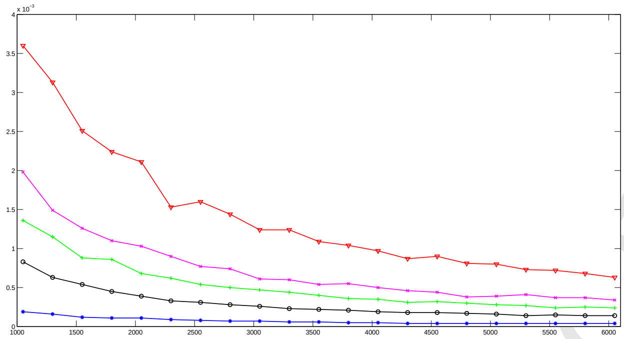

Figure 3: Empirical functional mean quadratic errors, based on $N = 1000$ generations of O.U. process, for different sample sizes (see horizontal axis), and for the values $\theta = 0.1$ (line with stars), $\theta = 0.4$ (line with circles), $\theta = 0.7$ (line with crosses), $\theta = 1$ (line with inclined crosses) and $\theta = 2$ (line with triangles)

3.2. Consistency of $\rho_{\widehat{\theta}_T} = \rho_{\widehat{\theta}_n}$ in $\mathcal{L}(H)$ and $\mathcal{L}(B)$

The strong-consistency of $\rho_{\widehat{\theta}_n}$ in $\mathcal{L}(H)$ is derived in Proposition 1 from the following almost surely upper bound

$$\|\rho_\theta - \rho_{\widehat{\theta}_n}\|^2_{\mathcal{L}(H)} \leq |\theta - \widehat{\theta}_n| h \sqrt{\frac{h}{3} + 1}. \tag{34}$$

Here, from $N = 1000$ simulations of the O.U. process on the interval $[0, T]$, with $T = nh = n = 200000 + (l-1)200000$ $(h = 1)$, for $l = 1, \ldots, 5$, the corresponding values of $\widehat{\theta}_T - \theta = \widehat{\theta}_n - \theta$ are computed, considering the cases $\theta = 0.4, 0.7, 1$. Table 2 shows the empirical probability of $\widehat{\theta}_T - \theta$ to lie within the band $\pm 3\sqrt{\frac{2\theta}{T}}$, for each one of sample sizes $T = 200000 + (l-1)200000$, $l = 1, \ldots, 5$, and cases $\theta = 0.4, 0.7, 1$, considered. It can be observed that for the sample sizes studied, in the case of $\theta = 1$, the empirical probabilities are equal to one. Thus, the almost surely convergence to zero of the upper bound (34) holds, with approximated convergence rate $\sqrt{T} = \sqrt{n}$. Note that, for the other two cases, $\theta = 0.4$ and $\theta = 0.7$, the empirical probabilities are also very close to one (see also Table 1 for smaller sample sizes, where we can also observe the empirical probabilities very close to one for the same band). In particular, Figure 4 displays the cases $\theta = 0.4$ (at the left-hand side) and $\theta = 1$ (at the right-hand side).

$T \backslash \theta$	0.4	0.7	1
200000	1	1	1
400000	1	1	1
600000	0.9988	1	1
800000	0.9988	0.9988	1
1000000	0.9977	1	1

Table 2: Empirical probability of $\widehat{\theta}_T - \theta$ to be within the band $\pm 3\sigma = \pm 3\sqrt{\frac{2\theta}{T}}$, from $N = 1000$ simulations of O.U. process over the interval $[0, T]$, with $T = n = 200000 + (l-1)200000$, $l = 1, \ldots, 5$, considering the cases $\theta = 0.4, 0.7, 1$

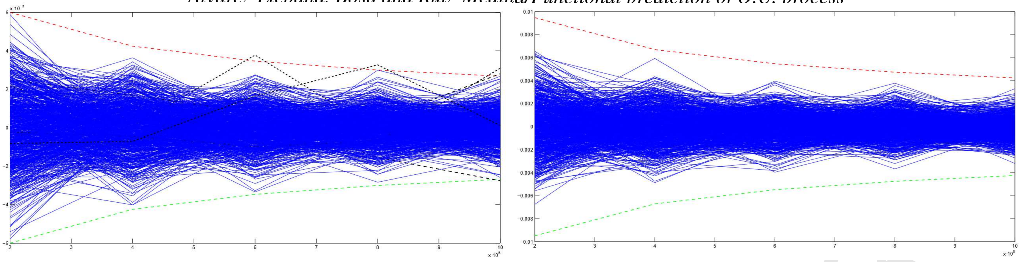

Figure 4: The values of $\widehat{\theta}_T - \theta$ are represented, corresponding to $N = 1000$ simulations of O.U. process over the interval $[0, T]$, with $T = n = 200000 + (l - 1)200000$, $l = 1, \dots, 5$, considering the cases $\theta = 0.4$ (at the left-hand side), and $\theta = 1$ (at right-hand side). The upper dotted line is $+3\sqrt{\frac{2\theta}{T}}$ and the lower dotted line is $-3\sqrt{\frac{2\theta}{T}}$

It can be observed from Table 2 that a better performance is obtained for the largest values of θ, which corresponds to the weakest dependent case. Furthermore, from the upper bound (27), the strong consistency of $\rho_{\widehat{\theta}_n}$ in $\mathcal{L}(B)$, with, as before, $B = \mathcal{C}([0, h])$, is also illustrated from the results displayed in Table 2 and Figure 4.

3.3. Consistency of the ARH(1) and ARB(1) plug-in predictors for the O.U. process

Let us now consider the derived upper bounds (22) and (29) in Corollaries 1 and 2, for the ARH(1) and ARB(1) predictors, respectively. From the generation of $N = 1000$ discrete realizations of the O.U. process over the interval $[0, T]$, for $T = n = 200000 + (l - 1)200000$, $l = 1, \dots, 5$, the upper bounds (22) and (29) are evaluated, for the cases $\theta = 0.4, 0.7, 1$. The following empirical probabilities for $\varepsilon_2 = 0.008$,

$$\widehat{P}^H_{\theta,N}(T) = 1 - \widehat{P}\left[|X_{n-1}(h)| |\theta - \widehat{\theta}_n| h \sqrt{\frac{h}{3} + 1} > \varepsilon \right] \tag{35}$$

$$N = 1000,\ T = n = 200000 + (l-1)200000,\ l = 1, \dots, 5,\ \theta = 0.4, 0.7, 1,$$

$$\widehat{P}^B_{\theta,N}(T) = 1 - \widehat{P}\left[|X_{n-1}(h)| |\theta - \widehat{\theta}_n| h > \varepsilon \right] \tag{36}$$

$$N = 1000,\ T = n = 200000 + (l-1)200000,\ l = 1, \dots, 5,\ \theta = 0.4, 0.7, 1,$$

are reflected in Table 3, for the Hilbert-valued (see (22)) and Banach-valued (see (29)) frameworks (see also Figure 5). It can be observed that the empirical probabilities are equal to one in both frameworks for the largest sample sizes, in any of the cases considered.

	Hilbert-valued case			Banach-valued case		
$T \backslash \theta$	0.4	0.7	1	0.4	0.7	1
200000	0.9800	0.9800	0.9800	0.9871	0.9906	0.9871
400000	0.9953	0.9953	0.9953	0.9965	0.9977	0.9977
600000	0.9988	0.9977	0.9988	0.9988	0.9988	1
800000	1	0.9988	0.9988	1	1	1
1000000	1	1	1	1	1	1

Table 3: Empirical probabilities (35) and (36), based on $N = 1000$ simulations of the O.U. process over the interval $[0, T]$, for $T = n = 200000 + (l - 1)200000$, $l = 1, \dots, 5$, considering the cases $\theta = 0.4, 0.7, 1$, and $\varepsilon_2 = 0.008$

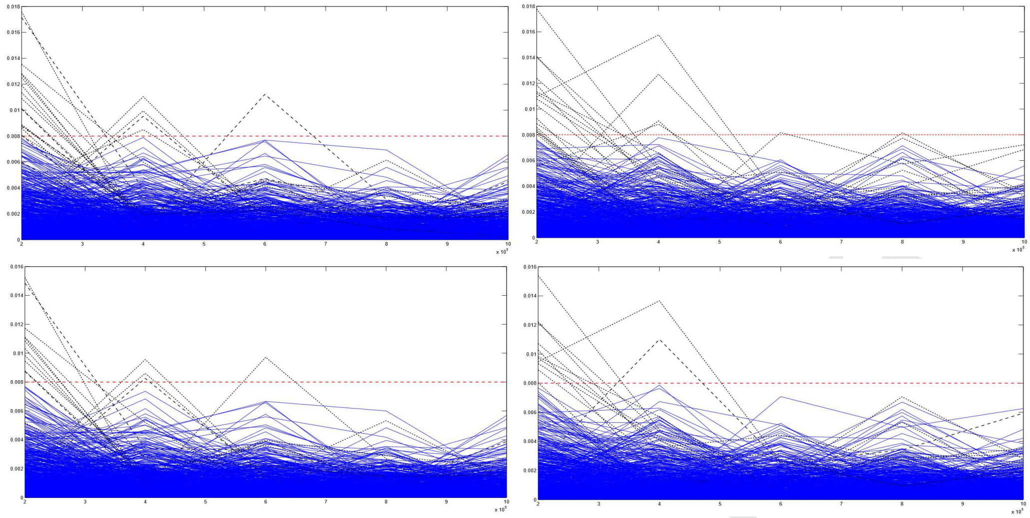

Figure 5: The values of $|X_{n-1}(h)||\theta - \widehat{\theta}_n|h\sqrt{\frac{h}{3}+1}$ (top) and $|X_{n-1}(h)||\theta - \widehat{\theta}_n|h$ (bottom) are represented, based on $N = 1000$ generations of O.U. process over the interval $[0, T]$, for $T = n = 200000 + (l - 1)200000$, $l = 1, \dots, 5$, against $\varepsilon = 0.008$ (horizontal dotted line), considering $\theta = 0.4$ (at the left-hand side) and $\theta = 1$ (at the right-hand side)

The strong-consistency of the MLE of θ and of the autocorrelation operator of the O.U. process, in Banach and Hilbert spaces, has been first illustrated. The almost surely rate of convergence to zero is shown as well. The numerical results on the consistency of the associated ARH(1) and ARB(1) plug-in predictors then follow, from the computation of the corresponding empirical probabilities for the derived upper bounds. Note that the numerical results displayed in Section 3 are obtained under generation of sample sizes ranging from 12000 up to a million of time instants, considering 1000 repetitions for each one of such sample sizes. In all these simulations performed, the discretization step size considered has been $\Delta t = 0.02$.

4. Final comments

The problem of functional prediction of the O.U. process could be of interest in several applied fields. For example, in finance, in the context of Vasicek model (see Vasicek (1977)) the results derived allow to predict the curve representing the interest rate over a temporal interval, in a consistent way. Note that, in this context, the ML estimate computed for parameter θ provides a consistent approximation of the speed reversion, which univocally determines the proposed functional predictor of the interest rate.

Summarizing, this paper addresses the problem of functional prediction of the O.U. process from ARH(1) and ARB(1) perspectives. Specifically, considering the O.U. process as an ARH(1) and an ARB(1) process, new results on strong consistency (almost surely convergence to the true parameter value), in the spaces $\mathcal{L}(H)$ and $\mathcal{L}(B)$ of the MLE of its autocorrelation operator are derived. Consistency results (convergence in probability to the true value) of the associated plug-in predictors are obtained as well. The numerical results shown, in addition, the normality and the asymptotic efficiency of the MLE of the scale parameter θ of the covariance function of the O.U. process.

Acknowledgements

This work has been supported in part by projects MTM2012-32674 (co-funded by European Regional Development Funds), and MTM2015–71839–P of the DGI, MINECO.

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