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Strongly consistent autoregressive predictors in abstract Banach spaces

María D. Ruiz-Medina *, Javier Álvarez-Liébana

Departamento de Estadística e Investigación Operativa, Universidad de Granada, Campus Fuente Nueva s/n, 18071 Granada, Spain

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ABSTRACT

This work derives new results on strong consistent estimation and prediction for autoregressive processes of order 1 in a separable Banach space B. The consistency results are obtained for the component-wise estimator of the autocorrelation operator in the norm of the space $\mathcal{L}(B)$ of bounded linear operators on B. The strong consistency of the associated plug-in predictor then follows in the B-norm. A Gelfand triple is defined through the Hilbert space constructed in Kuelbs' lemma (Kuelbs, 1970). A Hilbert–Schmidt embedding introduces the Reproducing Kernel Hilbert space (RKHS), generated by the autocovariance operator, into the Hilbert space conforming the Rigged Hilbert space structure. This paper extends the work of Bosq (2000) and Labbas and Mourid (2002).

1. Introduction

In the last few decades, there has been a growing interest in the statistical analysis of high-dimensional data from the Functional Data Analysis (FDA) perspective. The book by Ramsay and Silverman [31] provides an overview of FDA techniques adapted from the multivariate data context or specifically formulated for the FDA framework. The monograph by Hsing and Eubank [23] introduces functional analytical tools useful for the estimation of random elements in function spaces. The book by Horváth and Kokoszka [22] is mainly concerned with inference based on second order statistics; a central topic in this book is the analysis of functional data exhibiting dependent structures in time and space. The methodological survey paper by Cuevas [12] discusses central topics in FDA. Recent advances in the statistical analysis of high-dimensional data from the parametric, semiparametric and nonparametric FDA viewpoints are collected in the JMVA Special Issue by Goia and Vieu [21].

Linear time series models traditionally arise for processing temporal linear correlated data. In the FDA context, Bosq's monograph [9] introduces linear functional time series theory. The RKHS generated by the autocovariance operator plays a crucial role in the estimation approach presented therein. In particular, the eigenvectors of the autocovariance operator are considered for projection; see also [3]. Its empirical version is computed when they are unknown. The resulting plug-in predictor is obtained as a linear functional of the observations, based on the empirical approximation of the autocorrelation operator. This approach exploits the Hilbert space structure; its extension to the metric space context, particularly in the Banach space context, relies on a relationship (continuous embeddings) between the Banach space norm and the RKHS norm induced by the extended autocovariance operator. This is in contrast with the nonparametric regression approach,

* Corresponding author.
E-mail addresses: mruiz@ugr.es (M.D. Ruiz-Medina), javialvaliebana@ugr.es (J. Álvarez-Liébana).

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which considers semi-metric spaces; see, e.g., [17], where asymptotic normality is derived, in the regression model with functional response and covariates. In particular, a linear combination of the observed response values is considered, in the nonparametric local-weighting-based approach. Here, the weights are defined from an isotropic kernel depending on the metric or semi-metric of the space in which the regressors take their values; see, e.g., [18], and, in particular, [16] in the functional time series framework. However, the more flexible nonparametric approach also presents some computational drawbacks, requiring the resolution of several selection problems. For instance, a choice of smoothing parameter and the kernel involved in the definition of the weights should be performed. Real-valued covariates were incorporated in the semiparametric kernel-based proposal by Aneiros-Pérez and Vieu [5], which involves an extension to the functional partial linear time series framework; see also [4] on semi-functional partial linear regression. Goia and Vieu [20] also adopted a semiparametric approach in their formulation of a two-term Partitioned Functional Single Index Model. Geenens [19] exploited the alternative provided by semi-metrics to avoid the so-called *curse of dimensionality*.

In a parametric linear framework, functional time series models in Banach spaces were introduced in [26]. Strong mixing conditions and the absolute regularity of Banach-valued autoregressive processes were studied in [1]. Empirical estimators for Banach-valued autoregressive processes were discussed in [10] where, under some regularity conditions, and for the case of orthogonal innovations, the empirical mean was shown to be asymptotically optimal, with respect to almost sure convergence and convergence of order 2. The empirical autocovariance operator was also interpreted as a sample mean of an autoregressive process in a suitable space of linear operators. The extension of these results to the case of weakly dependent innovations was obtained in [14]. A strongly consistent sieve estimator of the autocorrelation operator of a Banach-valued autoregressive process was considered in [30]. Limit theorems for a seasonality estimator were given in [28] in the case of Banach autoregressive perturbations. Confidence regions were also obtained for the seasonality function in the Banach space of continuous functions. An approximation of Parzen's optimal predictor in the RKHS framework was used in [27] to predict temporal stochastic processes in Banach spaces. The existence and uniqueness of an almost surely strictly periodically correlated solution to the first order autoregressive model in Banach spaces was derived in [29]. Under some regularity conditions, limit results were obtained for AR$\mathcal{D}$(1) processes in [15], where $\mathcal{D}[0,1]$ denotes the Skorokhod space of right-continuous functions on [0, 1] having a left limit at all $t \in [0,1]$. Conditions for the existence of strictly stationary solutions of ARMA equations in Banach spaces with independent and identically distributed noise innovations can be found in [32].

In deriving strong consistency results for ARB(1) component-wise estimators and predictors, Bosq [9] restricted his attention to the case of the Banach space $\mathcal{C}[0,1]$ of continuous functions on [0, 1] with the sup norm. Labbas and Mourid [25] considered an ARB(1) context, for an arbitrary real separable Banach space B, under the construction of a Hilbert space $\widetilde{H}$, where B is continuously embedded as in Kuelbs' lemma [24]. Assuming the existence of a continuous extension to $\widetilde{H}$ of the autocorrelation operator $\rho \in \mathcal{L}(B)$, they proved the strong consistency of the formulated component-wise estimator of ρ and of its associated plug-in predictor in the norms of $\mathcal{L}(\widetilde{H})$ and $\widetilde{H}$, respectively.

The linear time series framework in Banach spaces studied here is motivated by the statistical analysis of temporal correlated functional data in nuclear spaces, arising notably in the observation of the solution to stochastic differential or fractional pseudodifferential equations; see, e.g., [7,8]. The scales of Banach spaces constituted by fractional Sobolev and Besov spaces play a central role in the context of nuclear spaces. Continuous (nuclear) embeddings usually connect the elements of these scales; see, e.g., [33]. In this paper a Rigged-Hilbert-Space structure is defined, involving the separable Hilbert space $\widetilde{H}$ appearing in the construction of Kuelbs' lemma [24]. A key assumption here is the existence of a continuous (Hilbert–Schmidt class) embedding introducing the RKHS associated with the extended autocovariance operator of the ARB(1) process, into the Hilbert space generating the Gelfand triple, equipped with a finer topology than the B-topology. Under this scenario, strong consistency results are derived in the space $\mathcal{L}(B)$ of bounded linear operators on B, considering an abstract separable Banach space framework.

This paper is structured as follows. Background material and notation are first given in Section 2. Section 3 states the basic assumptions and key lemmas which are then proved in Section 4. This paper's main strong consistency result is derived in Section 5, and examples are presented in Section 6. Closing comments are in Section 7. The results are illustrated numerically in an Online Supplement under the scenario described in Section 6.

2. Preliminaries

Let $(B, \|\cdot\|_B)$ be a real separable Banach space equipped with the norm $\|\cdot\|_B$ and let $\mathcal{L}_B^2(\Omega, \mathcal{A}, P)$ be the space of zero mean B-valued random variables X such that

$$\sqrt{\int_B \|X\|_B^2 \, dP} < \infty.$$

Let $X = \{X_n : n \in \mathbb{Z}\}$ be a zero mean B-valued stochastic process on the probability space $(\Omega, \mathcal{A}, P)$ such that, for all $n \in \mathbb{Z}$,

$$X_n = \rho(X_{n-1}) + \varepsilon_n, \tag{1}$$

where $\rho \in \mathcal{L}(B)$ denotes the autocorrelation operator of X; see [9]. In Eq. (1), the B-valued innovation process $\varepsilon = \{\varepsilon_n : n \in \mathbb{Z}\}$ on $(\Omega, \mathcal{A}, P)$ is assumed to be a strong white noise uncorrelated with the random initial condition. Thus ε is a zero mean Banach-valued stationary process with iid components and $\sigma_\varepsilon^2 = \mathrm{E}\{\|\varepsilon_n\|_B^2\} < \infty$ for all $n \in \mathbb{Z}$.

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Assume that there exists an integer $j_0 \in \{1, 2, \ldots\}$ such that

$$\|\rho^{j_0}\|_{\mathcal{L}(B)} < 1. \tag{2}$$

Then Eq. (1) admits a unique strictly stationary solution with $\sigma_X^2 = \mathrm{E}\{\|X_n\|_B^2\} < \infty$, i.e., belonging to $\mathcal{L}_B^2(\Omega, \mathcal{A}, P)$, given by $X_n = \sum_{j\in\mathbb{Z}} \rho^j(\varepsilon_{n-j})$, for each $n \in \mathbb{Z}$; see [9]. Under (2), the autocovariance operator C of an ARB(1) process X is defined from the autocovariance operator of $X_0 \in \mathcal{L}_B^2(\Omega, \mathcal{A}, P)$ as $C(x^\star) = \mathrm{E}\{x^\star(X_0)X_0\}$ for all $x^\star \in B^\star$, and the cross-covariance operator D is given by $D(x^\star) = \mathrm{E}\{x^\star(X_0)X_1\}$ for all $x^\star \in B^\star$. Since C is assumed to be a nuclear operator, then as per Eq. (6.24) on p. 156 of [9], there exists a sequence $\{x_j : j \geq 1\} \subset B$ such that, for every $x^\star \in B^\star$,

$$C(x^\star) = \sum_{j=1}^{\infty} x^\star(x_j)x_j, \quad \sum_{j=1}^{\infty} \|x_j\|_B^2 < \infty.$$

If D is also assumed to be a nuclear operator, then, as per Eq. (6.23) on p. 156 of [9], there exist sequences $\{y_j : j \geq 1\} \subset B$ and $\{x_j^{\star\star} : j \geq 1\} \subset B^{\star\star}$ such that, for every $x^\star \in B^\star$,

$$D(x^\star) = \sum_{j=1}^{\infty} x_j^{\star\star}(x^\star)y_j, \quad \sum_{j=1}^{\infty} \|x_j^{\star\star}\| \times \|y_j\| < \infty.$$

From Eqs. (6.45) and (6.58) on pp. 164–168 of [9], empirical estimators of C and D are respectively given, for all $x^\star \in B^\star$ and any integer $n \geq 2$, by

$$C_n(x^\star) = \frac{1}{n}\sum_{i=0}^{n-1} x^\star(X_i)(X_i), \quad D_n(x^\star) = \frac{1}{n-1}\sum_{i=0}^{n-2} x^\star(X_i)(X_{i+1}).$$

Lemma 2.1 in [24], recalled just below, plays a key role in our approach.

Lemma 1. *If B is a real separable Banach space with norm $\|\cdot\|_B$, then there exists an inner product $\langle\cdot,\cdot\rangle_{\widetilde{H}}$ on B such that the norm $\|\cdot\|_{\widetilde{H}}$, generated by $\langle\cdot,\cdot\rangle_{\widetilde{H}}$, is weaker than $\|\cdot\|_B$. The completion of B under the norm $\|\cdot\|_{\widetilde{H}}$ defines the Hilbert space $\widetilde{H}$, where B is continuously embedded.*

Denote by $\{x_n : n \geq 1\} \subset B$ a dense sequence in B and by $\{F_n : n \geq 1\} \subset B^\star$ a sequence of bounded linear functionals on B satisfying

$$F_n(x_n) = \|x_n\|_B, \quad \|F_n\| = 1, \tag{3}$$

such that, for all $x \in B$,

$$\|x\|_B = \sup_{n\geq 1} |F_n(x)|. \tag{4}$$

The inner product $\langle\cdot,\cdot\rangle_{\widetilde{H}}$, and associated norm, in Lemma 1, is defined, for all $x, y \in \widetilde{H}$, by

$$\langle x, y\rangle_{\widetilde{H}} = \sum_{n=1}^{\infty} t_n F_n(x)F_n(y),$$

while for all $x \in B$,

$$\|x\|_{\widetilde{H}}^2 = \sum_{n=1}^{\infty} t_n\{F_n(x)\}^2 \leq \|x\|_B^2, \tag{5}$$

where $\{t_n : n \geq 1\}$ is a sequence of positive numbers summing up to 1.

3. Main assumptions and preliminary results

In view of Lemma 1, for every $n \in \mathbb{Z}$, $X_n \in B \hookrightarrow \widetilde{H}$ satisfies almost surely,

$$X_n \underset{\widetilde{H}}{=} \sum_{j=1}^{\infty} \langle X_n, v_j\rangle_{\widetilde{H}} v_j,$$

where $\{v_j : j \geq 1\}$ is any orthonormal basis of $\widetilde{H}$. The trace auto-covariance operator

$$C = \mathrm{E}\left\{\left(\sum_{j=1}^{\infty} \langle X_n, v_j\rangle_{\widetilde{H}} v_j\right) \otimes \left(\sum_{j=1}^{\infty} \langle X_n, v_j\rangle_{\widetilde{H}} v_j\right)\right\}$$

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of the extended ARB(1) process is a trace operator on $\widetilde{H}$ admitting a diagonal spectral representation in terms of its eigenvalues $\{C_j : j \geq 1\}$ and eigenvectors $\{\phi_j : j \geq 1\}$ that provide an orthonormal system in $\widetilde{H}$. In what follows, the next identities in $\widetilde{H}$ will be considered, for the extended version of ARB(1) process X. For arbitrary $f, h \in \widetilde{H}$,

$$C(f) \underset{\widetilde{H}}{=} \sum_{j=1}^{\infty} C_j \langle f, \phi_j \rangle_{\widetilde{H}} \phi_j, \tag{6}$$

$$D(h) \underset{\widetilde{H}}{=} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle D(\phi_j), \phi_k \rangle_{\widetilde{H}} \langle h, \phi_j \rangle_{\widetilde{H}} \phi_k,$$

$$C_n(f) \underset{\widetilde{H}\ \text{a.s.}}{=} \sum_{j=1}^{n} C_{n,j} \langle f, \phi_{n,j} \rangle_{\widetilde{H}} \phi_{n,j}, \tag{7}$$

$$C_{n,j} \underset{\text{a.s.}}{=} \frac{1}{n} \sum_{i=0}^{n-1} X_{i,n,j}^2,\ X_{i,n,j} = \langle X_i, \phi_{n,j} \rangle_{\widetilde{H}},\ C_n(\phi_{n,j}) \underset{\widetilde{H}\ \text{a.s.}}{=} C_{n,j} \phi_{n,j},$$

$$D_n(h) \underset{\widetilde{H}\ \text{a.s.}}{=} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle D_n(\phi_{n,j}), \phi_{n,k} \rangle_{\widetilde{H}} \langle h, \phi_{n,j} \rangle_{\widetilde{H}} \phi_{n,k}, \tag{8}$$

where, for arbitrary integer $n \geq 2$, $\{\phi_{n,j} : j \geq 1\}$ is a complete orthonormal system in $\widetilde{H}$, and $C_{n,1} \geq \cdots \geq C_{n,n} \geq 0 = C_{n,n+1} = C_{n,n+2} = \cdots$

The following assumption plays a crucial role in the derivation of the main results in this paper.

Assumption A1. $\|X_0\|_B$ is almost surely bounded, and the eigenspace V_j associated with $C_j > 0$ in (6) is one-dimensional for every integer $j \geq 1$.

Under Assumption A1, we can define the following quantities:

$$a_1 = 2\sqrt{2}\, \frac{1}{C_1 - C_2}, \quad a_j = 2\sqrt{2}\, \max\left(\frac{1}{C_{j-1} - C_j}, \frac{1}{C_j - C_{j+1}} \right), \quad j \geq 2. \tag{9}$$

Remark 1. This assumption can be relaxed to consider multidimensional eigenspaces by redefining the quantities $a_1, a_2, \ldots$ as the quantities $c_1, c_2, \ldots$ given in Lemma 4.4 of [9].

Assumption A2. Let k_n be such that $C_{n,k_n} > 0$ a.s., and both $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$.

Remark 2. Consider

$$\Lambda_{k_n} = \sup_{j \in \{1,\ldots,k_n\}} (C_j - C_{j+1})^{-1}. \tag{10}$$

Then for sufficiently large n, we have

$$k_n < C_{k_n}^{-1} < \frac{1}{C_{k_n} - C_{k_n+1}} < a_{k_n} < \Lambda_{k_n} < \sum_{j=1}^{k_n} a_j.$$

Assumption A3. As $k \to \infty$,

$$\sup_{x \in B,\ \|x\|_B \leq 1} \left\| \rho(x) - \sum_{j=1}^{k} \langle \rho(x), \phi_j \rangle_{\widetilde{H}} \phi_j \right\|_B \to 0.$$

Assumption A4. The constants $\{C_j : j \geq 1\}$ are such that the inclusion of $\mathcal{H}(X)$ into $\widetilde{H}^\star$ is continuous, i.e., $\mathcal{H}(X) \hookrightarrow \widetilde{H}^\star$, where $\hookrightarrow$ denotes the continuous embedding.

Let us consider the closed subspace H of B with the norm induced by the inner product $\langle \cdot, \cdot \rangle_H$ defined as follows:

$$H = \left\{ x \in B : \sum_{n=1}^{\infty} \{F_n(x)\}^2 < \infty \right\}, \quad \forall_{f,g \in H}\ \ \langle f, g \rangle_H = \sum_{n=1}^{\infty} F_n(f) F_n(g). \tag{11}$$

Then H is continuously embedded into B and the following remark provides the isometric isomorphism established by the Riesz Representation Theorem between the spaces $\widetilde{H}$ and its dual $\widetilde{H}^\star$.

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Remark 3. Let $f^\star, g^\star \in \widetilde{H}^\star$, and $f, g \in \widetilde{H}$, such that, for every integer $n \geq 1$, consider $F_n(f^\star) = \sqrt{t_n}\, F_n(\widetilde{f})$, $F_n(g^\star) = \sqrt{t_n}\, F_n(\widetilde{g})$, and $F_n(\widetilde{f}) = \sqrt{t_n}\, F_n(f)$, $F_n(\widetilde{g}) = \sqrt{t_n}\, F_n(g)$ for certain $\widetilde{f}, \widetilde{g} \in H$. Then the following identities hold:

$$\langle f^\star, g^\star \rangle_{\widetilde{H}^\star} = \sum_{n=1}^{\infty} \frac{1}{t_n} F_n(f^\star) F_n(g^\star) = \sum_{n=1}^{\infty} \frac{1}{t_n} \sqrt{t_n}\sqrt{t_n} F_n(\widetilde{f}) F_n(\widetilde{g}) = \langle \widetilde{f}, \widetilde{g} \rangle_H = \sum_{n=1}^{\infty} t_n F_n(f) F_n(g) = \langle f, g \rangle_{\widetilde{H}}.$$

Lemma 2. *Under Assumption A4, the following continuous embeddings hold:*

$$\mathcal{H}(X) \hookrightarrow \widetilde{H}^\star \hookrightarrow B^\star \hookrightarrow H \hookrightarrow B \hookrightarrow \widetilde{H} \hookrightarrow \{\mathcal{H}(X)\}^\star, \tag{12}$$

where

$$\widetilde{H} = \left\{ x \in B : \sum_{n=1}^{\infty} t_n \{F_n(x)\}^2 < \infty \right\}, \quad \forall_{f,g \in \widetilde{H}} \ \langle f, g \rangle_{\widetilde{H}} = \sum_{n=1}^{\infty} t_n F_n(f) F_n(g),$$

$$H = \left\{ x \in B : \sum_{n=1}^{\infty} \{F_n(x)\}^2 < \infty \right\}, \quad \forall_{f,g \in H} \ \langle f, g \rangle_H = \sum_{n=1}^{\infty} F_n(f) F_n(g),$$

$$\widetilde{H}^\star = \left\{ x \in B : \sum_{n=1}^{\infty} \frac{1}{t_n} \{F_n(x)\}^2 < \infty \right\}, \quad \forall_{f,g \in \widetilde{H}^\star} \ \langle f, g \rangle_{\widetilde{H}^\star} = \sum_{n=1}^{\infty} F_n(f) F_n(g)/t_n,$$

$$\mathcal{H}(X) = \{x \in \widetilde{H};\ \langle C^{-1}(x), x \rangle_{\widetilde{H}} < \infty\}, \quad \forall_{f,g \in C^{1/2}(\widetilde{H})} \ \langle f, g \rangle_{\mathcal{H}(X)} = \langle C^{-1}(f), g \rangle_{\widetilde{H}},$$

$$\{\mathcal{H}(X)\}^\star = \{x \in \widetilde{H};\ \langle C(x), x \rangle_{\widetilde{H}} < \infty\}, \quad \forall_{f,g \in C^{-1/2}(\widetilde{H})} \ \langle f, g \rangle_{\{\mathcal{H}(X)\}^\star} = \langle C(f), g \rangle_{\widetilde{H}}.$$

Proof. Let us consider the following inequalities, valid for all $x \in B$ such that $x \in \widetilde{H}^\star$:

$$\|x\|_{\widetilde{H}} = \sqrt{\sum_{n=1}^{\infty} t_n \{F_n(x)\}^2} \leq \|x\|_B = \sup_{n \geq 1} |F_n(x)|,$$

$$\|x\|_B = \sup_{n \geq 1} |F_n(x)| \leq \sqrt{\sum_{n=1}^{\infty} \{F_n(x)\}^2} = \|x\|_H \leq \sum_{n=1}^{\infty} |F_n(x)| = \|x\|_{B^\star},$$

$$\|x\|_{B^\star} = \sum_{n=1}^{\infty} |F_n(x)| \leq \sqrt{\sum_{n=1}^{\infty} \frac{1}{t_n} \{F_n(x)\}^2} = \|x\|_{\widetilde{H}^\star}. \tag{13}$$

Under Assumption A4 (see also Remark 3), for every $f \in C^{1/2}(\widetilde{H}) = \mathcal{H}(X)$, we have

$$\|f\|_{\mathcal{H}(X)} = \sqrt{\langle C^{-1}(f), f \rangle_{\widetilde{H}}} \geq \|f\|_{\widetilde{H}^\star} = \sqrt{\sum_{n=1}^{\infty} \frac{1}{t_n} \{F_n(x)\}^2}. \tag{14}$$

From Eqs. (13) and (14), the inclusions in (12) are continuous. Thus the proof is complete. □

It is well known that the set $\{\phi_j : j \geq 1\}$ is also an orthogonal system in $\mathcal{H}(X)$. Furthermore, under Assumption A4, from Lemma 2, $\{\phi_j : j \geq 1\} \subset \mathcal{H}(X) \hookrightarrow \widetilde{H}^\star \hookrightarrow B^\star \hookrightarrow H$. Therefore, from (11), for every $j \geq 1$,

$$\|\phi_j\|_H^2 = \sum_{m=1}^{\infty} \{F_m(\phi_j)\}^2 < \infty. \tag{15}$$

The following assumption is now considered on the norm (15).

Assumption A5. The continuous embedding $i_{\mathcal{H}(X),H} : \mathcal{H}(X) \hookrightarrow H$ belongs to the Hilbert–Schmidt class, i.e., $\sum_{j=1}^{\infty} \|\phi_j\|_H^2 < \infty$. Let $\{F_m : m \geq 1\}$ be defined as in Lemma 1. Assumption A5 leads to

$$\sum_{j=1}^{\infty} \langle i_{\mathcal{H}(X),H}(\phi_j), i_{\mathcal{H}(X),H}(\phi_j) \rangle_H = \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \{F_m(\phi_j)\}^2 = \sum_{m=1}^{\infty} N_m < \infty, \tag{16}$$

where, in particular, from (16),

$$N_m = \sum_{j=1}^{\infty} \{F_m(\phi_j)\}^2 < \infty, \quad \sup_{m \geq 1} N_m = N < \infty, \quad V = \sup_{j \geq 1} \|\phi_j\|_B \leq \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \{F_m(\phi_j)\}^2 < \infty. \tag{17}$$

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The following preliminary results are deduced from Theorem 4.1 (pp. 98–99), Corollary 4.1 (pp. 100–101), and Theorem 4.8 (pp. 116–117) in [9].

Lemma 3. *Under Assumption A1, the following identities hold, for any standard $AR\widetilde{H}(1)$ process, e.g., the extension to $\widetilde{H}$ of $ARB(1)$ process X satisfying Eq. (1), as $n \to \infty$:*

$$\|C_n - C\|_{\mathcal{S}(\widetilde{H})} = \mathcal{O}[\{\ln(n)/n\}^{1/2}] \ a.s., \quad \|D_n - D\|_{\mathcal{S}(\widetilde{H})} = \mathcal{O}[\{\ln(n)/n\}^{1/2}] \ a.s. \tag{18}$$

Here, $\to_{a.s.}$ denotes almost surely convergence, and $\|\cdot\|_{\mathcal{S}(\widetilde{H})}$ is the norm in the Hilbert space $\mathcal{S}(\widetilde{H})$ of Hilbert–Schmidt operators on $\widetilde{H}$, i.e., the subspace of compact operators $\mathcal{A}$ such that $\sum_{j=1}^{\infty}\langle \mathcal{A}^{\star}\mathcal{A}(\varphi_j), \varphi_j\rangle_{\widetilde{H}} < \infty$, for any orthonormal basis $\{\varphi_j : j \geq 1\}$ of $\widetilde{H}$.

Lemma 4. *Under Assumption A1, let $\{C_j : j \geq 1\}$ and $\{C_{n,j} : j \geq 1\}$ in (6) and (7), respectively. Then, as $n \to \infty$, $\{n/\ln(n)\}^{1/2}\sup_{j\geq 1}|C_{n,j} - C_j| \to_{a.s.} 0$.*

The following lemma is Corollary 4.3 on p. 107 of [9].

Lemma 5. *Under Assumption A1, consider Λ_{k_n} in (10) satisfying $\Lambda_{k_n} = o[\{n/\ln(n)\}^{1/2}]$, as $n \to \infty$. Then, as $n \to \infty$,*

$$\sup_{j\in\{1,\ldots,k_n\}} \|\phi'_{n,j} - \phi_{n,j}\|_{\widetilde{H}} \to_{a.s.} 0,$$

where, for every integer $j \geq 1$ and $n \geq 2$, $\phi'_{n,j} = \operatorname{sgn}\langle \phi_{n,j}, \phi_j\rangle_{\widetilde{H}}\phi_j$, $\operatorname{sgn}\langle \phi_{n,j}, \phi_j\rangle_{\widetilde{H}} = \mathbf{1}_{\langle \phi_{n,j},\phi_j\rangle_{\widetilde{H}}\geq 0} - \mathbf{1}_{\langle \phi_{n,j},\phi_j\rangle_{\widetilde{H}}<0}$, with $\mathbf{1}$ denoting an indicator function.

An upper bound for $\|c\|_{B\times B} = \|\sum_{j=1}^{\infty} C_j\phi_j \otimes \phi_j\|_{B\times B}$ is obtained next.

Lemma 6. *Under Assumption A5, the following inequality holds:*

$$\|c\|_{B\times B} = \sup_{n,m\geq 1} |C(F_n)(F_m)| \leq N\,\|C\|_{\mathcal{L}(\widetilde{H})},$$

where N has been introduced in (17), $\mathcal{L}(\widetilde{H})$ denotes the space of bounded linear operators on $\widetilde{H}$, and $\|\cdot\|_{\mathcal{L}(\widetilde{H})}$ the usual uniform norm on such a space.

Let us consider the following notation:

$$c \underset{\widetilde{H}\otimes\widetilde{H}}{=} \sum_{j=1}^{\infty} C_j\phi'_{n,j}\otimes\phi'_{n,j} \underset{\widetilde{H}\otimes\widetilde{H}}{=} \sum_{j=1}^{\infty} C_j\phi_j\otimes\phi_j, \quad c_n \underset{\widetilde{H}\otimes\widetilde{H}}{=} \sum_{j=1}^{\infty} C_{n,j}\phi_{n,j}\otimes\phi_{n,j},$$

$$c - c_n \underset{\widetilde{H}\otimes\widetilde{H}}{=} \sum_{j=1}^{\infty} C_j\phi'_{n,j}\otimes\phi'_{n,j} - \sum_{j=1}^{\infty} C_{n,j}\phi_{n,j}\otimes\phi_{n,j}. \tag{19}$$

Remark 4. From Lemma 3, for sufficiently large n, there exist positive constants K_1 and K_2 such that, for all $\varphi \in \widetilde{H}$,

$$K_1\langle C(\varphi), \varphi\rangle_{\widetilde{H}} \leq \langle C_n(\varphi), \varphi\rangle_{\widetilde{H}} \leq K_2\langle C(\varphi), \varphi\rangle_{\widetilde{H}}.$$

In particular, for every $x \in \mathcal{H}(X) = C^{1/2}(\widetilde{H})$, considering n sufficiently large, we find

$$\frac{1}{K_1}\langle C^{-1}(x), x\rangle_{\widetilde{H}} \geq \langle C_n^{-1}(x), x\rangle_{\widetilde{H}} \geq \frac{1}{K_2}\langle C^{-1}(x), x\rangle_{\widetilde{H}} \quad \Leftrightarrow \quad \frac{1}{K_1}\|x\|^2_{\mathcal{H}(X)} \geq \langle C_n^{-1}(x), x\rangle_{\widetilde{H}} \geq \frac{1}{K_2}\|x\|^2_{\mathcal{H}(X)}. \tag{20}$$

Eq. (20) means that, for sufficiently large n, the norm of the RKHS $\mathcal{H}(X)$ of X is equivalent to the norm of the RKHS generated by C_n, with spectral kernel c_n given in (19).

Lemma 7. *Under Assumptions A1– A5, consider Λ_{k_n} in (10) satisfying*

$$\sqrt{k_n}\Lambda_{k_n} = o\left\{\sqrt{n/\ln(n)}\right\} \tag{21}$$

as $n \to \infty$, where k_n has been introduced in Assumption A2. The following almost sure inequality then holds:

$$\|c - c_n\|_{B\times B} \leq \max(N, \sqrt{N})\left[\|C - C_n\|_{\mathcal{L}(\widetilde{H})} + 2\max\left(\sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}}, \sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}}\right)\left[\sup_{\ell\geq 1}\sup_{m\geq 1}|F_\ell(\phi'_{n,m})|\right]\right.$$
$$\left.\times\sqrt{k_n 8\Lambda^2_{k_n}\|C_n - C\|^2_{\mathcal{L}(\widetilde{H})} + \sum_{m=k_n+1}^{\infty}\|\phi_{n,m} - \phi'_{n,m}\|^2_{\widetilde{H}}}\right].$$

Therefore, $\|c - c_n\|_{B\times B} \to_{a.s.} 0$ as $n \to \infty$.

Lemma 8. *For a standard ARB(1) process satisfying Eq. (1), under Assumptions A1– A5, consider Λ_{k_n} in (10) such that*

$$C_{k_n}^{-1}\sqrt{k_n}\Lambda_{k_n} = o\left\{\sqrt{n/\ln(n)}\right\} \tag{22}$$

as $n \to \infty$, where k_n has been introduced in Assumption A2. The following inequality then is established

$$\begin{aligned}\sup_{j\in\{1,\ldots,k_n\}} \left\|\phi_{n,j}-\phi'_{n,j}\right\|_B \leq \frac{2}{C_{k_n}}\Bigg[&\max(N,\sqrt{N})\Bigg[\|C-C_n\|_{\mathcal{L}(\widetilde{H})}+2\max\left(\sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}},\sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}}\right)\left\{\sup_{\ell\geq 1}\sup_{m\geq 1}|F_\ell(\phi'_{n,m})|\right\}\\ &\times\sqrt{k_n 8\Lambda_{k_n}^2\|C_n-C\|^2_{\mathcal{L}(\widetilde{H})}+\sum_{m=k_n+1}^{\infty}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}}\Bigg]\\ &+\sup_{j\in\{1,\ldots,k_n\}}\|\phi_{n,j}-\phi'_{n,j}\|_{\widetilde{H}}N\|C\|_{\mathcal{S}(\widetilde{H})}+V\|C-C_n\|_{\mathcal{S}(\widetilde{H})}\Bigg] \quad a.s.\end{aligned} \tag{23}$$

Therefore, under (22), $\sup_{j\in\{1,\ldots,k_n\}}\|\phi_{n,j}-\phi'_{n,j}\|_B \to_{a.s.} 0$, *as* $n\to\infty$.

Lemma 9. *Under Assumption A3, if* $\sum_{j=1}^{k_n}\|\phi_{n,j}-\phi'_{n,j}\|_B \to_{a.s.} 0$ *as* $n\to\infty$, *then also*

$$\sup_{x\in B,\ \|x\|_B\leq 1}\left\|\rho(x)-\sum_{j=1}^{k_n}\langle\rho(x),\phi_{n,j}\rangle_{\widetilde{H}}\phi_{n,j}\right\|_B \to_{a.s.} 0. \tag{24}$$

Remark 5. Under the conditions of Lemma 8, Eq. (24) holds as soon as

$$C_{k_n}^{-1}k_n^{3/2}\Lambda_{k_n} = o\left\{\sqrt{n/\ln(n)}\right\}.$$

Let us know consider the projection operators defined, for all $x\in B\subset\widetilde{H}$, by

$$\widetilde{\Pi}^{k_n}(x)=\sum_{j=1}^{k_n}\langle x,\phi_{n,j}\rangle_{\widetilde{H}}\phi_{n,j},\quad \Pi^{k_n}(x)=\sum_{j=1}^{k_n}\langle x,\phi'_{n,j}\rangle_{\widetilde{H}}\phi'_{n,j}. \tag{25}$$

Remark 6. Under the conditions of Remark 5, let

$$\widetilde{\Pi}^{k_n}\rho\widetilde{\Pi}^{k_n}=\sum_{j=1}^{k_n}\sum_{p=1}^{k_n}\langle\rho(\phi_{n,j}),\phi_{n,p}\rangle_{\widetilde{H}}\phi_{n,j}\otimes\phi_{n,p}.$$

Then, as $n\to\infty$,

$$\sup_{x\in B,\ \|x\|_B\leq 1}\left\|\rho(x)-\sum_{j=1}^{k_n}\sum_{p=1}^{k_n}\langle x,\phi_{n,j}\rangle_{\widetilde{H}}\langle\rho(\phi_{n,j}),\phi_{n,p}\rangle_{\widetilde{H}}\phi_{n,p}\right\|_B \to_{a.s.} 0.$$

4. Proofs of the lemmas

4.1. Proof of Lemma 6

Applying the Cauchy–Schwarz inequality, we have, for all integers $k,\ell\geq 1$,

$$\begin{aligned}|C(F_k,F_\ell)| &= \left|\sum_{j=1}^{\infty}C_jF_k(\phi_j)F_\ell(\phi_j)\right| \leq \sqrt{\sum_{j=1}^{\infty}C_j\{F_k(\phi_j)\}^2\sum_{p=1}^{\infty}C_p\{F_\ell(\phi_p)\}^2}\\ &\leq \sup_{j\geq 1}|C_j|\sqrt{\sum_{j=1}^{\infty}\{F_k(\phi_j)\}^2\sum_{p=1}^{\infty}\{F_\ell(\phi_p)\}^2} = \sup_{j\geq 1}|C_j|\sqrt{N_kN_\ell},\end{aligned} \tag{26}$$

where $\{F_n : n\geq 1\}$ has been introduced in Eq. (3), and satisfies (4)–(5). Under Assumption A5, from Eq. (17),

$$\|c\|_{B\times B} = \sup_{k,\ell\geq 1}|C(F_k,F_\ell)| \leq \sup_{k,\ell\geq 1}\sup_{j\geq 1}|C_j|\sqrt{N_kN_\ell} = N\sup_{j\geq 1}|C_j| = N\|C\|_{\mathcal{L}(\widetilde{H})}.$$

This completes the proof. □

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4.2. Proof of Lemma 7

First observe that

$$|C - C_n(F_k)(F_\ell)| = \left|\sum_{j=1}^{\infty} C_j F_k(\phi'_{n,j}) F_\ell(\phi'_{n,j}) - C_{n,j} F_k(\phi_{n,j}) F_\ell(\phi_{n,j})\right|$$
$$\leq \sum_{j=1}^{\infty} |C_j||F_k(\phi'_{n,j})||F_\ell(\phi'_{n,j}) - F_\ell(\phi_{n,j})| + \sup_{j\geq 1}|C_j - C_{n,j}||F_k(\phi'_{n,j})F_\ell(\phi_{n,j})|$$
$$+ |C_{n,j}F_\ell(\phi_{n,j})||F_k(\phi'_{n,j}) - F_k(\phi_{n,j})|.$$

Next note that the right-hand side is bounded above by

$$\sqrt{\sum_{j=1}^{\infty} C_j\{F_k(\phi'_{n,j})\}^2 \sum_{j=1}^{\infty} C_j\{F_\ell(\phi'_{n,j}) - F_\ell(\phi_{n,j})\}^2}$$
$$+ \sup_{j\geq 1}|C_j - C_{n,j}| \sqrt{\sum_{j=1}^{\infty}\{F_k(\phi'_{n,j})\}^2 \sum_{j=1}^{\infty}\{F_\ell(\phi_{n,j})\}^2}$$
$$+ \sqrt{\sum_{j=1}^{\infty} C_{n,j}\{F_\ell(\phi_{n,j})\}^2 \sum_{j=1}^{\infty} C_{n,j}\{F_k(\phi'_{n,j}) - F_k(\phi_{n,j})\}^2},$$

and that this upper bound is itself smaller than

$$\sqrt{N_k}\sqrt{\sum_{j=1}^{\infty} C_j\{F_\ell(\phi'_{n,j}) - F_\ell(\phi_{n,j})\}^2} + \sup_{j\geq 1}|C_j - C_{n,j}|\sqrt{N_k}\sqrt{N_\ell} + \sqrt{N_\ell}\sqrt{\sum_{j=1}^{\infty} C_{n,j}\{F_k(\phi'_{n,j}) - F_k(\phi_{n,j})\}^2}.$$

Now, the latter expression can itself be bounded above by

$$\max(N, \sqrt{N})\left[\sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}\sum_{j=1}^{\infty}\{F_\ell(\phi'_{n,j} - \phi_{n,j})\}^2} + \|C - C_n\|_{\mathcal{L}(\widetilde{H})} + \sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}\sum_{j=1}^{\infty}\{F_k(\phi'_{n,j} - \phi_{n,j})\}^2}\right]$$

and hence, a fortiori, also by

$$\max(N, \sqrt{N})\left[\|C - C_n\|_{\mathcal{L}(\widetilde{H})} + \sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\{F_\ell(\phi'_{n,m})\}^2\{\langle\phi'_{n,j}, \phi'_{n,m}\rangle_{\widetilde{H}} - \langle\phi_{n,j}, \phi'_{n,m}\rangle_{\widetilde{H}}\}^2}\right.$$
$$\left. + \sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\{F_k(\phi'_{n,m})\}^2\{\langle\phi'_{n,j}, \phi'_{n,m}\rangle_{\widetilde{H}} - \langle\phi_{n,j}, \phi'_{n,m}\rangle_{\widetilde{H}}\}^2}\right].$$

The latter expression can be rewritten as

$$\max(N, \sqrt{N})\left[\|C - C_n\|_{\mathcal{L}(\widetilde{H})} + \sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}\sum_{m=1}^{\infty}\{F_\ell(\phi'_{n,m})\}^2\sum_{j=1}^{\infty}\{\langle\phi'_{n,j}, \phi'_{n,m}\rangle_{\widetilde{H}} - \langle\phi_{n,j}, \phi'_{n,m}\rangle_{\widetilde{H}}\}^2}\right.$$
$$\left. + \sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}\sum_{m=1}^{\infty}\{F_k(\phi'_{n,m})\}^2\sum_{j=1}^{\infty}\{\langle\phi'_{n,j}, \phi'_{n,m}\rangle_{\widetilde{H}} - \langle\phi_{n,j}, \phi'_{n,m}\rangle_{\widetilde{H}}\}^2}\right],$$

or equivalently as

$$\max(N, \sqrt{N})\left[\|C - C_n\|_{\mathcal{L}(\widetilde{H})} + \sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}\sum_{m=1}^{\infty}\{F_\ell(\phi'_{n,m})\}^2\sum_{j=1}^{\infty}\{\langle\phi_{n,j}, \phi_{n,m}\rangle_{\widetilde{H}} - \langle\phi_{n,j}, \phi'_{n,m}\rangle_{\widetilde{H}}\}^2}\right.$$
$$\left. + \sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}\sum_{m=1}^{\infty}\{F_k(\phi'_{n,m})\}^2\sum_{j=1}^{\infty}\{\langle\phi_{n,j}, \phi_{n,m}\rangle_{\widetilde{H}} - \langle\phi_{n,j}, \phi'_{n,m}\rangle_{\widetilde{H}}\}^2}\right],$$

and also as

$$\max(N,\sqrt{N})\Bigg[\|C-C_n\|_{\mathcal{L}(\widetilde{H})}+\sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}\sum_{m=1}^{\infty}\{F_\ell(\phi'_{n,m})\}^2\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}}$$
$$+\sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}\sum_{m=1}^{\infty}\{F_k(\phi'_{n,m})\}^2\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}}\Bigg].$$

It is now easy to see that the latter expression is bounded above by

$$\max(N,\sqrt{N})\Bigg\{\|C-C_n\|_{\mathcal{L}(\widetilde{H})}+\sup_{m\geq 1}|F_\ell(\phi'_{n,m})|\sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}\sum_{m=1}^{\infty}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}}$$
$$+\sup_{m\geq 1}|F_k(\phi'_{n,m})|\sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}\sum_{m=1}^{\infty}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}}\Bigg\}.$$

Recapping, we can then conclude that $|C-C_n(F_k)(F_\ell)|$ is smaller than the above term, and hence also

$$|C-C_n(F_k)(F_\ell)|\leq \max(N,\sqrt{N})\Bigg[\|C-C_n\|_{\mathcal{L}(\widetilde{H})}+\max\left(\sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}},\sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}}\right)$$
$$\left\{\sup_{m\geq 1}|F_\ell(\phi'_{n,m})|+\sup_{m\geq 1}|F_k(\phi'_{n,m})|\right\}\sqrt{\sum_{m=1}^{\infty}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}}\Bigg]. \tag{27}$$

Note that, under Assumption A5, from Eq. (16), we have, for every integer $k\geq 1$,

$$\sup_{m\geq 1}|F_k(\phi'_{n,m})|<\infty. \tag{28}$$

Thus, considering k_n, as given in Assumption A2, from Lemma 5, under (21), applying Eq. (18), as $n\to\infty$, we get

$$\sum_{m=1}^{\infty}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}=\sum_{m=1}^{k_n}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}+\sum_{m=k_n+1}^{\infty}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}$$
$$\leq k_n\sup_{1\leq m\leq k_n}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}+\sum_{m=k_n+1}^{\infty}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}$$
$$\leq k_n 8\Lambda^2_{k_n}\|C_n-C\|^2_{\mathcal{L}(\widetilde{H})}+\sum_{m=k_n+1}^{\infty}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}} \tag{29}$$
$$\leq k_n 8\Lambda^2_{k_n}\|C_n-C\|^2_{\mathcal{S}(\widetilde{H})}+\sum_{m=k_n+1}^{\infty}\|\phi_{n,m}-\phi'_{n,m}\|^2_{\widetilde{H}}\to_{\text{a.s.}}0. \tag{30}$$

From Eqs. (27)–(30), considering that under Assumption A5, we have

$$\sup_{k\geq 1}\sup_{m\geq 1}|F_k(\phi'_{n,m})|<\infty,$$

we conclude that $\|c-c_n\|_{B\times B}=\sup_{k,\ell\geq 1}|(C-C_n)(F_k)(F_\ell)|\to_{\text{a.s.}}0$ as $n\to\infty$. □

4.3. Proof of Lemma 8

Let us first consider the following string of almost sure equalities:

$$C_{n,j}(\phi_{n,j}-\phi'_{n,j})=C_n(\phi_{n,j})-C_{n,j}(\phi'_{n,j})=(C_n-C)(\phi_{n,j})+C(\phi_{n,j}-\phi'_{n,j})+(C_j-C_{n,j})\phi'_{n,j}. \tag{31}$$

From Eq. (31), keeping in mind Assumption A2, we can write

$$\left\|\phi_{n,j}-\phi'_{n,j}\right\|_B\leq\frac{1}{C_{n,j}}\|(C_n-C)(\phi_{n,j})\|_B+\frac{1}{C_{n,j}}\|C(\phi_{n,j}-\phi'_{n,j})\|_B+\frac{1}{C_{n,j}}\|(C_j-C_{n,j})\phi'_{n,j}\|_B$$
$$=\frac{1}{C_{n,j}}(S_1+S_2+S_3),\quad \text{a.s.} \tag{32}$$

For sufficiently large n, from Lemmas 6–7, applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned} S_1 = \| (C_n - C)(\phi_{n,j})\|_B &= \sup_{m\geq 1}\left|\sum_{k=1}^{\infty} C_{n,k}F_m(\phi_{n,k})\langle \phi_{n,k}, \phi_{n,j}\rangle_{\widetilde{H}} - \sum_{k=1}^{\infty} C_k F_m(\phi'_{n,k})\langle \phi'_{n,k}, \phi_{n,j}\rangle_{\widetilde{H}}\right| \\ &= \sup_{m\geq 1}\left|\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} t_\ell F_\ell(\phi_{n,j})\{C_{n,k}F_m(\phi_{n,k})F_\ell(\phi_{n,k}) - C_k F_m(\phi'_{n,k})F_\ell(\phi'_{n,k})\}\right| \\ &= \sup_{m\geq 1}\left|\sum_{\ell=1}^{\infty} t_\ell F_\ell(\phi_{n,j})\sum_{k=1}^{\infty} C_{n,k}F_m(\phi_{n,k})F_\ell(\phi_{n,k}) - C_k F_m(\phi'_{n,k})F_\ell(\phi'_{n,k})\right| \end{aligned}$$

for every integer $j \geq 1$. Now the right-hand side can be bounded above by

$$\sup_{m\geq 1}\sqrt{\sum_{\ell=1}^{\infty} t_\ell\{F_\ell(\phi_{n,j})\}^2}\times\sqrt{\sum_{\ell=1}^{\infty} t_\ell\left\{\sum_{k=1}^{\infty} C_{n,k}F_m(\phi_{n,k})F_\ell(\phi_{n,k}) - C_k F_m(\phi'_{n,k})F_\ell(\phi'_{n,k})\right\}^2},$$

which in turn, is smaller than

$$\|\phi_{n,j}\|_{\widetilde{H}}\sqrt{\sum_{\ell=1}^{\infty} t_\ell}\sup_{m,\ell}\left|\sum_{k=1}^{\infty} C_{n,k}F_m(\phi_{n,k})F_\ell(\phi_{n,k}) - C_k F_m(\phi'_{n,k})F_\ell(\phi'_{n,k})\right| = \|c_n - c\|_{B\times B}.$$

Thus S_1 is smaller than the latter expression, from which we can then deduce that

$$\begin{aligned} S_1 \leq \max(N,\sqrt{N})\Bigg[&\|C - C_n\|_{\mathcal{L}(\widetilde{H})} + 2\max\left(\sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}}, \sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}}\right)\left\{\sup_{\ell\geq 1}\sup_{m\geq 1}|F_\ell(\phi'_{n,m})|\right\} \\ &\times\sqrt{k_n 8\Lambda_{k_n}^2\|C_n - C\|^2_{\mathcal{L}(\widetilde{H})} + \sum_{m=k_n+1}^{\infty}\|\phi_{n,m} - \phi'_{n,m}\|^2_{\widetilde{H}}}\Bigg]. \end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned} S_2 = \|C(\phi_{n,j} - \phi'_{n,j})\|_B &= \sup_{m\geq 1}\left|\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} t_\ell C_k F_m(\phi'_{n,k})F_\ell(\phi'_{n,k})F_\ell(\phi_{n,j} - \phi'_{n,j})\right| \\ &\leq \sup_{m\geq 1}\sqrt{\sum_{\ell=1}^{\infty} t_\ell\{F_\ell(\phi_{n,j} - \phi'_{n,j})\}^2}\sqrt{\sum_{\ell=1}^{\infty} t_\ell\left\{\sum_{k=1}^{\infty} C_k F_m(\phi'_{n,k})F_\ell(\phi'_{n,k})\right\}^2} \\ &\leq \|\phi_{n,j} - \phi'_{n,j}\|_{\widetilde{H}}\sup_{m,\ell\geq 1}\left|\sum_{k=1}^{\infty} C_k F_m(\phi'_{n,k})F_\ell(\phi'_{n,k})\right| \\ &= \|\phi_{n,j} - \phi'_{n,j}\|_{\widetilde{H}}\times\|c\|_{B\times B} \leq \|\phi_{n,j} - \phi'_{n,j}\|_{\widetilde{H}}\times N\times\|C\|_{\mathcal{S}(\widetilde{H})} \text{ a.s.} \end{aligned} \tag{34}$$

Moreover, we see that under Assumption A5,

$$S_3 \leq \sup_{j\geq 1}|C_j - C_{n,j}|\left\|\phi'_{n,j}\right\|_B \leq V\|C - C_n\|_{\mathcal{L}(\widetilde{H})} \leq V\|C - C_n\|_{\mathcal{S}(\widetilde{H})} \text{ a.s.} \tag{35}$$

In addition, it follows from Lemma 3 that $\|C_n - C\|_{\mathcal{S}(\widetilde{H})} \to_{\text{a.s.}} 0$, and, from Lemma 4, $C_{n,j} \to_{\text{a.s.}} C_j$ as $n \to \infty$. For $\varepsilon = C_{k_n}/2$, we can thus find n_0 such that for $n \geq n_0$,

$$\begin{aligned} &\|C_n - C\|_{\mathcal{L}(\widetilde{H})} \leq \varepsilon = C_{k_n}/2 \text{ a.s.}, \quad |C_{n,k_n} - C_{k_n}| \leq \widetilde{\varepsilon} \leq \|C_n - C\|_{\mathcal{L}(\widetilde{H})}, \\ &C_{n,k_n} \geq C_{k_n} - \widetilde{\varepsilon} \geq C_{k_n} - \|C_n - C\|_{\mathcal{L}(\widetilde{H})} \geq C_{k_n} - C_{k_n}/2 \geq C_{k_n}/2. \end{aligned} \tag{36}$$

From Eqs. (32)–(35), and for n large enough to ensure that Eq. (36) holds, the following almost surely inequalities are then satisfied for all $j \in \{1, \ldots, k_n\}$:

$$\sup_{j\in\{1,\ldots,k_n\}} \|\phi_{n,j} - \phi'_{n,j}\|_B \leq \frac{1}{C_{n,k_n}} \Bigg[\max(N, \sqrt{N}) \Bigg[\|C - C_n\|_{\mathcal{L}(\widetilde{H})} + 2\max\left(\sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}}, \sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}}\right) \left\{ \sup_{\ell\geq 1} \sup_{m\geq 1} |F_\ell(\phi'_{n,m})| \right\} \times \sqrt{k_n 8 \Lambda_{k_n}^2 \|C_n - C\|^2_{\mathcal{L}(\widetilde{H})} + \sum_{m=k_n+1}^{\infty} \|\phi_{n,m} - \phi'_{n,m}\|^2_{\widetilde{H}}} \Bigg] + \sup_{j\in\{1,\ldots,k_n\}} \|\phi_{n,j} - \phi'_{n,j}\|_{\widetilde{H}} N \|C\|_{\mathcal{S}(\widetilde{H})} + V\|C - C_n\|_{\mathcal{S}(\widetilde{H})} \Bigg],$$

and, from Eq. (36), the right-hand term is bounded above by

$$\frac{2}{C_{k_n}} \Bigg[\max(N, \sqrt{N}) \Bigg[\|C - C_n\|_{\mathcal{L}(\widetilde{H})} + 2\max\left(\sqrt{\|C\|_{\mathcal{L}(\widetilde{H})}}, \sqrt{\|C_n\|_{\mathcal{L}(\widetilde{H})}}\right) \left\{ \sup_{\ell\geq 1} \sup_{m\geq 1} |F_\ell(\phi'_{n,m})| \right\} \times \sqrt{k_n 8 \Lambda_{k_n}^2 \|C_n - C\|^2_{\mathcal{L}(\widetilde{H})} + \sum_{m=k_n+1}^{\infty} \|\phi_{n,m} - \phi'_{n,m}\|^2_{\widetilde{H}}} \Bigg] + \sup_{j\in\{1,\ldots,k_n\}} \|\phi_{n,j} - \phi'_{n,j}\|_{\widetilde{H}} N \|C\|_{\mathcal{S}(\widetilde{H})} + V\|C - C_n\|_{\mathcal{S}(\widetilde{H})} \Bigg] \text{ a.s.}$$

Hence, Eq. (23) holds. The almost sure convergence to zero directly follows from Lemma 3, under (22). □

4.4. Proof of Lemma 9

The following identities are considered:

$$\sum_{j=1}^{k_n} \langle \rho(x), \phi_{n,j}\rangle_{\widetilde{H}} \phi_{n,j} - \sum_{j=1}^{k_n} \langle \rho(x), \phi'_{n,j}\rangle_{\widetilde{H}} \phi'_{n,j} = \sum_{j=1}^{k_n} \langle \rho(x), \phi_{n,j}\rangle_{\widetilde{H}} (\phi_{n,j} - \phi'_{n,j}) + \sum_{j=1}^{k_n} \langle \rho(x), \phi_{n,j} - \phi'_{n,j}\rangle_{\widetilde{H}} \phi'_{n,j}. \tag{37}$$

From Eq. (37), applying the Cauchy–Schwarz inequality, under Assumption A3, we find

$$\sup_{x\in B,\ \|x\|_B\leq 1} \left\| \sum_{j=1}^{k_n} \langle \rho(x), \phi_{n,j}\rangle_{\widetilde{H}} \phi_{n,j} - \sum_{j=1}^{\infty} \langle \rho(x), \phi'_{n,j}\rangle_{\widetilde{H}} \phi'_{n,j} \right\|_B$$

$$\leq \sup_{x\in B,\ \|x\|_B\leq 1} \sum_{j=1}^{k_n} \|\rho(x)\|_{\widetilde{H}} \times \|\phi_{n,j}\|_{\widetilde{H}} \times \|\phi_{n,j} - \phi'_{n,j}\|_B + \|\rho(x)\|_{\widetilde{H}} \times \|\phi_{n,j} - \phi'_{n,j}\|_{\widetilde{H}} \times \|\phi'_{n,j}\|_B$$

$$+ \sup_{x\in B,\ \|x\|_B\leq 1} \left\| \sum_{j=k_n+1}^{\infty} \langle \rho(x), \phi'_{n,j}\rangle_{\widetilde{H}} \phi'_{n,j} \right\|_B.$$

Now the right-hand side can be bounded above by

$$\sup_{x\in B,\ \|x\|_B\leq 1} \|\rho(x)\|_{\widetilde{H}} \left(\sum_{j=1}^{k_n} \|\phi_{n,j} - \phi'_{n,j}\|_B + \|\phi_{n,j} - \phi'_{n,j}\|_B \sup_{j\geq 1} \|\phi'_{n,j}\|_B \right) + \sup_{x\in B,\ \|x\|_B\leq 1} \left\| \sum_{j=k_n+1}^{\infty} \langle \rho(x), \phi'_{n,j}\rangle_{\widetilde{H}} \phi'_{n,j} \right\|_B,$$

and the latter expression is smaller than

$$\sup_{x\in B,\ \|x\|_B\leq 1} \|\rho\|_{\mathcal{L}(\widetilde{H})} \|x\|_{\widetilde{H}} (1 + V) \sum_{j=1}^{k_n} \|\phi_{n,j} - \phi'_{n,j}\|_B + \sup_{x\in B,\ \|x\|_B\leq 1} \left\| \sum_{j=k_n+1}^{\infty} \langle \rho(x), \phi'_{n,j}\rangle_{\widetilde{H}} \phi'_{n,j} \right\|_B.$$

One can then conclude because this last expression is itself smaller than

$$\|\rho\|_{\mathcal{L}(\widetilde{H})} (1 + V) \sum_{j=1}^{k_n} \|\phi_{n,j} - \phi'_{n,j}\|_B + \sup_{x\in B,\ \|x\|_B\leq 1} \left\| \sum_{j=k_n+1}^{\infty} \langle \rho(x), \phi'_{n,j}\rangle_{\widetilde{H}} \phi'_{n,j} \right\|_B,$$

which tends a.s. to 0 as $n \to \infty$. This concludes the proof. □

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5. ARB(1) estimation and prediction: Strong consistency results

For every $x \in B \subset \widetilde{H}$, the following component-wise estimator $\widetilde{\rho}_{k_n}$ of ρ will be considered:

$$\widetilde{\rho}_{k_n}(x) = \left(\widetilde{\Pi}^{k_n} D_n C_n^{-1} \widetilde{\Pi}^{k_n}\right)(x) = \left\{ \sum_{j=1}^{k_n} \frac{1}{C_{n,j}} \langle x, \phi_{n,j}\rangle_{\widetilde{H}} \widetilde{\Pi}^{k_n} D_n(\phi_{n,j}) \right\},$$

where $\widetilde{\Pi}^{k_n}$ has been introduced in Eq. (25), and C_n, $C_{n,j}$, $\phi_{n,j}$ and D_n have been defined in Eqs. (7)–(8), respectively.

Theorem 1. *As before, let X be a standard ARB(1) process. Under the conditions of Lemmas 8 and 9 (see Remark 5), for all $\eta > 0$,*

$$\Pr\left(\|\widetilde{\rho}_{k_n} - \rho\|_{\mathcal{L}(B)} \geq \eta\right) \leq \mathcal{K} \exp(-n\eta^2/Q_n),$$

where

$$Q_n = \mathcal{O}\left\{ \left(C_{k_n}^{-1} k_n \sum_{j=1}^{k_n} a_j \right)^2 \right\}$$

as $n \to \infty$. Therefore, if

$$k_n C_{k_n}^{-1} \sum_{j=1}^{k_n} a_j = o\left\{\sqrt{n/\ln(n)}\right\},$$

as $n \to \infty$, then also, $\|\widetilde{\rho}_{k_n} - \rho\|_{\mathcal{L}(B)} \to_{a.s} 0$.

Proof. For every $x \in B$, such that $\|x\|_B \leq 1$, applying the triangle inequality, under Assumptions A1 and A2,

$$\|\widetilde{\Pi}^{k_n} D_n C_n^{-1} \widetilde{\Pi}^{k_n}(x) - \widetilde{\Pi}^{k_n} \rho \widetilde{\Pi}^{k_n}(x)\|_B \leq \|\widetilde{\Pi}^{k_n}(D_n - D) C_n^{-1} \widetilde{\Pi}^{k_n}(x)\|_B + \|\widetilde{\Pi}^{k_n}(DC_n^{-1} - \rho)\widetilde{\Pi}^{k_n}(x)\|_B = S_1(x) + S_2(x). \quad (38)$$

Under the conditions assumed in Lemma 9, considering inequality (36),

$$\begin{aligned} S_1(x) = \|\widetilde{\Pi}^{k_n}(D_n - D)C_n^{-1}\widetilde{\Pi}^{k_n}(x)\|_B &\leq \left\| C_{n,k_n}^{-1} \sum_{j=1}^{k_n}\sum_{p=1}^{k_n} \langle x, \phi_{n,j}\rangle_{\widetilde{H}} \langle (D_n - D)(\phi_{n,j}), \phi_{n,p}\rangle_{\widetilde{H}} \phi_{n,p} \right\|_B \\ &\leq \left|C_{n,k_n}^{-1}\right| \sum_{j=1}^{k_n}\sum_{p=1}^{k_n} |\langle x, \phi_{n,j}\rangle_{\widetilde{H}}| \times |\langle (D_n - D)(\phi_{n,j}), \phi_{n,p}\rangle_{\widetilde{H}}| \times \|\phi_{n,p}\|_B \leq 2C_{k_n}^{-1} k_n \|D_n - D\|_{\mathcal{L}(\widetilde{H})} \sum_{p=1}^{k_n} \|\phi_{n,p}\|_B \\ &\leq 2VC_{k_n}^{-1} k_n^2 \|D_n - D\|_{\mathcal{S}(\widetilde{H})}. \end{aligned} \quad (39)$$

Furthermore, applying the triangle inequality, we find

$$\begin{aligned} S_2(x) &= \|\widetilde{\Pi}^{k_n}(DC_n^{-1} - \rho)\widetilde{\Pi}^{k_n}(x)\|_B \\ &\leq \|\widetilde{\Pi}^{k_n} DC_n^{-1} \widetilde{\Pi}^{k_n}(x) - \widetilde{\Pi}^{k_n} DC^{-1} \Pi^{k_n}(x)\|_B + \|\widetilde{\Pi}^{k_n} DC^{-1}\Pi^{k_n}(x) - \widetilde{\Pi}^{k_n}\rho\widetilde{\Pi}^{k_n}(x)\|_B = S_{21}(x) + S_{22}(x). \end{aligned} \quad (40)$$

Under Assumptions A1–A2, C^{-1} and C_n^{-1} are bounded on the subspaces generated by $\{\phi_j : j \in \{1, \ldots, k_n\}\}$ and $\{\phi_{n,j} : j \in \{1, \ldots, k_n\}\}$, respectively. Consider now

$$\begin{aligned} S_{21}(x) &= \|\widetilde{\Pi}^{k_n} DC_n^{-1}\widetilde{\Pi}^{k_n}(x) - \widetilde{\Pi}^{k_n} DC^{-1}\Pi^{k_n}(x)\|_B \\ &= \left\| \sum_{j=1}^{k_n}\sum_{p=1}^{k_n} \frac{1}{C_{n,j}} \langle x, \phi_{n,j} - \phi'_{n,j}\rangle_{\widetilde{H}} \langle D(\phi_{n,j}), \phi_{n,p}\rangle_{\widetilde{H}} \phi_{n,p} + \sum_{j=1}^{k_n}\sum_{p=1}^{k_n} \left(\frac{1}{C_{n,j}} - \frac{1}{C_j}\right) \langle x, \phi'_{n,j}\rangle_{\widetilde{H}} \langle D(\phi_{n,j}), \phi_{n,p}\rangle_{\widetilde{H}} \phi_{n,p} \right. \\ &\qquad \left. + \sum_{j=1}^{k_n}\sum_{p=1}^{k_n} \frac{1}{C_j} \langle x, \phi'_{n,j}\rangle_{\widetilde{H}} \langle D(\phi_{n,j} - \phi'_{n,j}), \phi_{n,p}\rangle_{\widetilde{H}} \phi_{n,p} \right\|_B. \end{aligned}$$

Note that

$$\begin{aligned} S_{21}(x) \leq & \sum_{j=1}^{k_n}\sum_{p=1}^{k_n} \left|\frac{1}{C_{n,k_n}}\right| \|\phi_{n,j} - \phi'_{n,j}\|_{\widetilde{H}} \times \|D\|_{\mathcal{L}(\widetilde{H})} \times \|\phi_{n,p}\|_B \\ & + \left|\frac{1}{C_{n,j}} - \frac{1}{C_j}\right| \|D\|_{\mathcal{L}(\widetilde{H})} \times \|\phi_{n,p}\|_B + \left|\frac{1}{C_j}\right| \|D\|_{\mathcal{L}(\widetilde{H})} \times \|\phi_{n,j} - \phi'_{n,j}\|_{\widetilde{H}} \times \|\phi_{n,p}\|_B. \end{aligned} \quad (41)$$

From Lemma 4.3 on p. 104 of [9], for every integer $j \geq 1$, under Assumption A1,

$$\|\phi_{n,j} - \phi'_{n,j}\|_{\widetilde{H}} \leq a_j \|C_n - C\|_{\mathcal{L}(\widetilde{H})}, \tag{42}$$

where a_j has been introduced in (9). Then, in Eq. (41), considering again inequality (36), keeping in mind that $C_j^{-1} \leq a_j$, we obtain

$$S_{21}(x) \leq 5C_{k_n}^{-1} \sum_{p=1}^{k_n} \|\phi_{n,p}\|_B \times \|D\|_{\mathcal{L}(\widetilde{H})} \times \|C_n - C\|_{\mathcal{L}(\widetilde{H})} \sum_{j=1}^{k_n} a_j \leq 5Vk_n C_{k_n}^{-1} \|D\|_{\mathcal{L}(\widetilde{H})} \times \|C_n - C\|_{\mathcal{S}(\widetilde{H})} \sum_{j=1}^{k_n} a_j. \tag{43}$$

Applying again the triangle and Cauchy–Schwarz inequalities, we deduce from (42) that

$$\begin{aligned}
S_{22} &= \|\widetilde{\Pi}^{k_n} DC^{-1} \Pi^{k_n}(x) - \widetilde{\Pi}^{k_n} \rho \widetilde{\Pi}^{k_n}(x)\|_B \\
&= \left\| \sum_{j=1}^{k_n} \sum_{p=1}^{k_n} \langle x, \phi'_{n,j} - \phi_{n,j} \rangle_{\widetilde{H}} \langle \rho(\phi'_{n,j}), \phi_{n,p} \rangle_{\widetilde{H}} \phi_{n,p} + \langle x, \phi_{n,j} \rangle_{\widetilde{H}} \langle \rho(\phi'_{n,j} - \phi_{n,j}), \phi_{n,p} \rangle_{\widetilde{H}} \phi_{n,p} \right\|_B \\
&\leq \sum_{j=1}^{k_n} \sum_{p=1}^{k_n} \|x\|_{\widetilde{H}} \times \|\phi'_{n,j} - \phi_{n,j}\|_{\widetilde{H}} \times \|\rho\|_{\mathcal{L}(\widetilde{H})} \times \|\phi'_{n,j}\|_{\widetilde{H}} \times \|\phi_{n,p}\|_{\widetilde{H}} \times \|\phi_{n,p}\|_B \\
&\qquad + \|x\|_{\widetilde{H}} \times \|\phi_{n,j}\|_{\widetilde{H}} \times \|\rho\|_{\mathcal{L}(\widetilde{H})} \times \|\phi'_{n,j} - \phi_{n,j}\|_{\widetilde{H}} \times \|\phi_{n,p}\|_{\widetilde{H}} \times \|\phi_{n,p}\|_B \\
&\leq 2\|\rho\|_{\mathcal{L}(\widetilde{H})} \times \|C_n - C\|_{\mathcal{S}(\widetilde{H})} \left(\sum_{p=1}^{k_n} \|\phi_{n,p}\|_B \right) \left(\sum_{j=1}^{k_n} a_j \right) \\
&\leq 2V\|\rho\|_{\mathcal{L}(\widetilde{H})} \times \|C_n - C\|_{\mathcal{S}(\widetilde{H})} k_n \sum_{j=1}^{k_n} a_j.
\end{aligned} \tag{44}$$

From Eqs. (38)–(44),

$$\begin{aligned}
&\sup_{x \in B,\ \|x\|_B \leq 1} \|\widetilde{\Pi}^{k_n} D_n C_n^{-1} \widetilde{\Pi}^{k_n}(x) - \widetilde{\Pi}^{k_n} \rho \widetilde{\Pi}^{k_n}(x)\|_B \leq 2VC_{k_n}^{-1} k_n^2 \|D_n - D\|_{\mathcal{S}(\widetilde{H})} \\
&+ \|C_n - C\|_{\mathcal{S}(\widetilde{H})} 2Vk_n \sum_{j=1}^{k_n} a_j (5/2C_{k_n}^{-1} \|D\|_{\mathcal{L}(\widetilde{H})} + \|\rho\|_{\mathcal{L}(\widetilde{H})}).
\end{aligned} \tag{45}$$

From Eq. (45), applying now Theorem 4.2 (p. 99) and Theorem 4.8 (p. 116) in [9], we get, for any $\eta > 0$,

$$\begin{aligned}
&\Pr\left(\sup_{x \in B,\ \|x\|_B \leq 1} \|\widetilde{\Pi}^{k_n} D_n C_n^{-1} \widetilde{\Pi}^{k_n}(x) - \widetilde{\Pi}^{k_n} \rho \widetilde{\Pi}^{k_n}(x)\|_B > \eta \right) \\
&\leq \Pr\left\{ \sup_{x \in B,\ \|x\|_B \leq 1} S_1(x) > \eta \right\} + \Pr\left\{ \sup_{x \in B,\ \|x\|_B \leq 1} S_{21}(x) + S_{22}(x) > \eta \right\} \\
&\leq \Pr\left(\|D_n - D\|_{\mathcal{S}(\widetilde{H})} > \frac{\eta}{2VC_{k_n}^{-1} k_n^2} \right) + \Pr\left(\|C_n - C\|_{\mathcal{S}(\widetilde{H})} > \frac{\eta}{2Vk_n \sum_{j=1}^{k_n} a_j \left[5/2C_{k_n}^{-1} \|D\|_{\mathcal{L}(\widetilde{H})} + \|\rho\|_{\mathcal{L}(\widetilde{H})} \right]} \right) \\
&\leq 8 \exp\left[- \frac{n\eta^2}{\left(2VC_{k_n}^{-1} k_n^2\right)^2 \left[\gamma + \delta\{\eta/(2VC_{k_n}^{-1} k_n^2)\} \right]} \right] + 4\exp(-n\eta^2/Q_n),
\end{aligned} \tag{46}$$

with γ and δ being positive numbers, depending on ρ and P_{ε_0}, respectively, introduced in Theorems 4.2 and 4.8 of [9]. Here,

$$Q_n = 4V^2 k_n^2 \left(\sum_{j=1}^{k_n} a_j \right)^2 \left\{ 5/2C_{k_n}^{-1} \|D\|_{\mathcal{L}(\widetilde{H})} + \|\rho\|_{\mathcal{L}(\widetilde{H})} \right\}^2 \times \left\{ \alpha_1 + \beta_1 \frac{\eta}{2Vk_n \sum_{j=1}^{k_n} a_j \left(5/2C_{k_n}^{-1} \|D\|_{\mathcal{L}(\widetilde{H})} + \|\rho\|_{\mathcal{L}(\widetilde{H})} \right)} \right\}, \tag{47}$$

where again α_1 and β_1 are positive constants depending on ρ and P_{ε_0}, respectively. From Eqs. (46)–(47), we see that if $k_n C_{k_n}^{-1} \sum_{j=1}^{k_n} a_j = o\left\{\sqrt{n/\ln(n)}\right\}$ as $n \to \infty$, then, the Borel–Cantelli Lemma, and Lemma 9 lead to the desired almost sure convergence to zero (see also Remarks 5–6). □

Corollary 1. *Under the conditions of Theorem 1,* $\|\widetilde{\rho}_{k_n}(X_n) - \rho(X_n)\|_B \to_{a.s.} 0$ *as* $n \to \infty$.

The proof is straightforward from Theorem 1 because $\|\widetilde{\rho}_{k_n}(X_n) - \rho(X_n)\|_B \leq \|\widetilde{\rho}_{k_n} - \rho\|_{\mathcal{L}(B)} \|X_0\|_B \to_{a.s} 0$ as $n \to \infty$ under Assumption A1.

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6. Examples: Wavelets in Besov and Sobolev spaces

It is well known that wavelets provide orthonormal bases of $L^2(\mathbb{R})$, and unconditional bases for several function spaces including Besov spaces, $B^s_{p,q}$ with $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Sobolev or Hölder spaces constitute interesting particular cases of Besov spaces; see, e.g., [33].

Consider now orthogonal wavelets on the interval $[0, 1]$. Adapting wavelets to a finite interval requires some modifications as described in [11]. Let $s > 0$ for an $([s] + 1)$-regular Multiresolution Analysis (MRA) of $L^2[0, 1]$, where $[\cdot]$ stands for the integer part. The father and mother wavelets φ and ψ are such that $\varphi, \psi \in \mathcal{C}^{[s]+1}[0, 1]$. Also φ and its derivatives, up to order $[s] + 1$, have a fast decay; see Corollary 5.2 in [13].

Given J such that $2^J \geq 2([s] + 1)$, the construction in [11] starts from a finite set of 2^J scaling functions $\{\varphi_{J,k} : k \in \{0, \ldots, 2^J - 1\}\}$. For each $j \geq J$, a set 2^j wavelet functions $\{\psi_{j,k} : k \in \{0, \ldots, 2^j - 1\}\}$ are also considered. The collection of these functions $\{\varphi_{J,k} : k \in \{0, \ldots, 2^J - 1\}\}$ and $\{\psi_{j,k} : k \in \{0, \ldots, 2^j - 1\}\}$ with $j \geq J$ form a complete orthonormal system of $L^2[0, 1]$. The associated reconstruction formula is given, for all $t \in [0, 1]$ and $f \in L^2[0, 1]$, by

$$f(t) = \sum_{k=0}^{2^J-1} \alpha^f_{J,k} \varphi_{J,k}(t) + \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \beta^f_{j,k} \psi_{j,k}(t), \tag{48}$$

where

$$\alpha^f_{J,k} = \int_0^1 f(t) \overline{\varphi_{J,k}(t)} dt$$

for all $k \in \{0, \ldots, 2^J - 1\}$ and

$$\beta^f_{j,k} = \int_0^1 f(t) \overline{\psi_{j,k}(t)} dt$$

for all $k \in \{0, \ldots, 2^j - 1\}$ with $j \geq J$.

The Besov spaces $B^s_{p,q}[0, 1]$ can be characterized in terms of wavelets coefficients. Specifically, denote by $\mathcal{S}'$ the dual of $\mathcal{S}$, the Schwarz space. Then $f \in \mathcal{S}'$ belongs to $B^s_{p,q}[0, 1]$ for some $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$ if and only if

$$\|f\|^s_{p,q} \equiv \|\varphi * f\|_p + \left\{ \sum_{j=1}^{\infty} \left(2^{js} \|\psi_j * f\|_p\right)^q \right\}^{1/q} < \infty. \tag{49}$$

For $\beta > 1/2$, consider a self-adjoint positive operator $\mathcal{T} : H_2^{-\beta}[0, 1] \longrightarrow H_2^{\beta}[0, 1]$ on $L^2[0, 1]$ belonging to the unit ball of trace operators on $L^2[0, 1]$. Assume that $\mathcal{T} : H_2^{-\beta}[0, 1] \longrightarrow H_2^{\beta}[0, 1]$ and $\mathcal{T}^{-1} : H_2^{\beta}[0, 1] \longrightarrow H_2^{-\beta}[0, 1]$ are bounded linear operators. In particular, there exists an orthonormal basis $\{v_k : k \geq 1\}$ of $L^2[0, 1]$ such that, for every $\ell \geq 1$, $\mathcal{T}(v_\ell) = t_\ell v_\ell$, with $\sum_{\ell \geq 1} t_\ell = 1$.

In what follows, consider $\{v_\ell : \ell \geq 1\}$ to be an orthonormal wavelet basis, and define the kernel t of $\mathcal{T}$, for all $s, t \in [0, 1]$, by

$$t(s, t) = \frac{1}{2^J} \sum_{k=0}^{2^J-1} \varphi_{J,k}(s) \varphi_{J,k}(t) + \frac{2^{2\beta} - 1}{2^{2\beta(1-J)}} \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} 2^{-2j\beta} \psi_{j,k}(s) \psi_{j,k}(t).$$

In Lemma 1, $(F_{\mathbf{m}}) = \{F^{\varphi}_{J,k} : k \in \{0, \ldots, 2^J - 1\}\} \cup \{F^{\psi}_{j,k} : k \in \{0, \ldots, 2^j - 1, j \geq J\}\}$ are then defined as follows:

$$\begin{aligned} F^{\varphi}_{J,k} &= \varphi_{J,k}, \quad k \in \{0, \ldots, 2^J - 1\} \\ F^{\psi}_{j,k} &= \psi_{j,k}, \quad k \in \{0, \ldots, 2^j - 1\}, \quad j \geq J. \end{aligned} \tag{50}$$

Furthermore, the sequence $(t_{\mathbf{m}}) = \{t^{\varphi}_{J,k} : k \in \{0, \ldots, 2^J - 1\}\} \cup \{t^{\psi}_{j,k} : k \in \{0, \ldots, 2^j - 1\}, j \geq J\}$ involved in the definition of the inner product in $\widetilde{H}$, is given by

$$\begin{aligned} t^{\varphi}_{J,k} &= \frac{1}{2^J}, \quad k \in \{0, \ldots, 2^{J-1}\}. \\ t^{\psi}_{j,k} &= \frac{2^{2\beta} - 1}{2^{2\beta(1-J)}} 2^{-2j\beta}, \quad k \in \{0, \ldots, 2^{j-1}\}, \quad j \geq J. \end{aligned} \tag{51}$$

In view of Proposition 2.1 in [6], the choice (50)–(51) of $(F_{\mathbf{m}})$ and $(t_{\mathbf{m}})$ leads to the definition of $\widetilde{H} = \{H_2^{\beta}[0, 1]\}^{\star} = H_2^{-\beta}[0, 1]$, constituted by the restriction to $[0, 1]$ of the tempered distributions $g \in \mathcal{S}'(\mathbb{R})$, such that $(I - \Delta)^{-\beta/2} g \in L^2(\mathbb{R})$, with $(I - \Delta)^{-\beta/2}$ denoting the Bessel potential of order β; see [33].

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Now define $B = B^0_{\infty,\infty}([0,1],)$ and $B^\star = B^0_{1,1}[0,1]$. From Eq. (49), the corresponding norms, in terms of the discrete wavelet transform introduced in Eq. (48), are respectively given, for every $f \in B$, by

$$\|f\|_B = \sup\left\{|\alpha^f_{J,k}|, k \in \{0,\ldots,2^{J-1}\}; |\beta^f_{j,k}|, k \in \{0,\ldots,2^j-1\}, j \geq J\right\}$$

and for every $g \in B^\star$, by

$$\|g\|_{B^\star} = \sum_{k=0}^{2^J-1}|\alpha^g_{J,k}| + \sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1}|\beta^g_{j,k}|.$$

Therefore,

$$B^\star = B^0_{1,1}[0,1] \hookrightarrow H = L^2[0,1] \hookrightarrow B = B^0_{\infty,\infty} \hookrightarrow \widetilde{H} = H_2^{-\beta}[0,1].$$

Also, for $\beta > 1/2$, $\widetilde{H}^\star = H^\beta[0,1] \hookrightarrow B^\star = B^0_{1,1}[0,1]$.

For $\gamma > 2\beta$, consider the operator $C = (I-\Delta)^{-\gamma}$, i.e., given by the $2\gamma/\beta$ power of the Bessel potential of order β, restricted to $L^2[0,1]$. From spectral theorems on spectral calculus stated, e.g., in [33], we have that, for every $f \in C^{1/2}\{H^{-\beta}[0,1]\}$,

$$\begin{aligned}\|f\|^2_{\mathcal{H}(X)} = \langle C^{-1}(f), f\rangle_{H^{-\beta}[0,1]} &= \langle (I-\Delta)^{-\beta/2}\{C^{-1}(f)\}, (I-\Delta)^{-\beta/2}(f)\rangle_{L^2[0,1]} \\ &= \sum_{j=1}^{\infty} f_j^2\lambda_j\{(I-\Delta)^{(\gamma-\beta)}\} \geq \sum_{j=1}^{\infty} f_j^2\lambda_j\{(I-\Delta)^{\beta}\} = \|f\|^2_{H^\beta[0,1]} = \|f\|^2_{\widetilde{H}^\star},\end{aligned}$$

where, for every integer $j \geq 1$,

$$f_j = \int_0^1 f(s)\phi_j(s)ds,$$

with $\{\phi_j : j \geq 1\}$ denoting the eigenvectors of the Bessel potential $(I-\Delta)^{-\beta/2}$ of order β, restricted to $L^2[0,1]$, and $\{\lambda_j\{(I-\Delta)^{\gamma-\beta}\} : j \geq 1\}$ being the eigenvalues of $(I-\Delta)^{-\beta}C^{-1}$ on $L^2[0,1]$. Thus, Assumption A4 holds. Furthermore, from Embedding Theorems between fractional Sobolev spaces (see [33]), Assumption A5 also holds, under the condition $\gamma > 2\beta > 1$, considering $H = L^2[0,1]$.

7. Final comments

Section 6 illustrates the motivation of the presented approach in relation to functional prediction in nuclear spaces. Specifically, the current literature on ARB(1) prediction has been developed for $B = \mathcal{C}[0,1]$, the space of continuous functions on $[0,1]$, with the sup norm (see, e.g., [2,9]), and $B = \mathcal{D}[0,1]$ consisting of the right-continuous functions on $[0,1]$ having a left limit at every $t \in [0,1]$, with the Skorokhod topology; see, e.g., [15]. This paper provides a more flexible framework, where functional prediction can be performed, in a consistent way, for instance, in nuclear spaces, as follows from the continuous inclusions showed in Section 6. Note that the two above-referred usual Banach spaces, $\mathcal{C}[0,1]$ and $\mathcal{D}[0,1]$, are included in the Banach space B considered in Section 6; see the Online Supplement for the results of a simulation study.

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Appendix A. Supplementary simulation study

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jmva.2018.08.001.

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