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# A note on strong-consistency of componentwise ARH(1) predictors

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ABSTRACT

New results on strong-consistency in the trace operator norm are obtained, in the parameter estimation of an autoregressive Hilbertian process of order one (ARH(1) process). Additionally, a strongly-consistent diagonal componentwise estimator of the autocorrelation operator is derived, based on its empirical singular value decomposition.



## 1. Introduction

There exists an extensive literature on Functional Data Analysis (FDA) techniques. In the past few years, the primary focus of FDA was mainly on independent and identically distributed (i.i.d.) functional observations. The classical book by Ramsay and Silverman (2005) provides a wide overview on FDA techniques (e.g., regression, principal components analysis, linear modeling, canonical correlation analysis, curve registration, and principal differential analysis, etc.). An introduction to nonparametric statistical approaches for FDA can be found in Ferraty and Vieu (2006). We also refer to the recent monograph by Hsing and Eubank (2015), where the usual functional analytical tools in FDA are introduced, addressing several statistical and estimation problems for random elements in function spaces. Special attention is paid to the monograph by Horváth and Kokoszka (2012) covering functional inference based on second order statistics.

We refer the reader to the methodological survey paper by Cuevas (2014), covering nonparametric techniques and discussing central topics in FDA. Recent advances on statistics in high/infinite dimensional spaces are collected in the IWFOS'14 Special Issue published in the Journal of Multivariate Analysis (see Goia and Vieu (2016) who summarized its contributions, providing a brief discussion on the current literature).

A central issue in FDA is to take into account the temporal dependence of the observations. Although the literature on scalar and vector time series is huge, there are relatively few contributions dealing with functional time series, and, in general,

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with dependent functional data. For instance, Part III (Chapters 13–18) of the monograph by Horváth and Kokoszka (2012) is devoted to this issue, including topics related to functional time series (in particular, the functional autoregressive model), and the statistical analysis of spatially distributed functional data. The moment-based notion of weak dependence introduced in Hörmann and Kokoszka (2010) is also accommodated to the statistical analysis of functional time series. This notion does not require the specification of a data model, and can be used to study the properties of many nonlinear sequences (see e.g., Hörmann (2008) and Berkes et al. (2011), for recent applications).

This paper adopts the methodological approach presented in Bosq (2000) for functional time series. That monograph studies the theory of linear functional time series, both in Hilbert and Banach spaces, focusing on the functional autoregressive model. Several authors have studied the asymptotic properties of componentwise estimators of the autocorrelation operator of an ARH(1) process, and of the associated plug-in predictors. We refer to Guillas (2001) and Mas (1999, 2004, 2007), where the efficiency, consistency and asymptotic normality of these estimators are addressed, in a parametric framework (see also Álvarez-Liébana et al. (2016), on estimation of the Ornstein–Uhlenbeck processes in Banach spaces, and Álvarez-Liébana et al. (2017), on weak consistency in the Hilbert–Schmidt operator norm of componentwise estimators). Particularly, strong-consistency in the norm of the space of bounded linear operators was derived in Bosq (2000). In the derivation of these results, the autocorrelation operator is usually assumed to be a Hilbert–Schmidt operator, when the eigenvectors of the autocovariance operator are unknown. This paper proves that, under basically the same setting of conditions as in the cited papers, the componentwise estimator of the autocorrelation operator proposed in Bosq (2000), based on the empirical eigenvectors of the autocovariance operator, is also strongly-consistent in the Hilbert–Schmidt and trace operator norms.

The dimension reduction problem constitutes also a central topic in the parametric, nonparametric and semiparametric FDA statistical frameworks. Special attention to this topic has been paid, for instance, in the context of functional regression with functional response and functional predictors (see, for example, Ferraty et al. (2012), where asymptotic normality is derived, and, Ferraty et al. (2002), in the functional time series framework). In the semiparametric and nonparametric estimation techniques, a kernel-based formulation is usually adopted. Real-valued covariates were incorporated in the novel semiparametric kernel-based proposal by Aneiros-Pérez and Vieu (2008), providing an extension to the functional partial linear time series framework (see also Aneiros-Pérez and Vieu (2006)). Motivated by spectrometry applications, a two-terms Partitioned Functional Single Index Model is introduced in Goia and Vieu (2015), in a semiparametric framework. In the ARH(1) process framework, the present paper provides a new diagonal componentwise estimator of the autocorrelation operator, based on its empirical singular value decomposition. Its strong-consistency is proved as well. The diagonal design leads to an important dimension reduction, going beyond the usual isotropic restriction on the kernels involved in the approximation of the regression operator (respectively, autocorrelation operator), in the nonparametric framework. Recently, Petrovich and Reimherr (2017) address the dimension reduction provided by the functional principal component projections in the general case when eigenvalues can be repeated, instead of the classical assumptions that their multiplicity should be one.

The outline of the paper is the following. Section 2 introduces basic definitions and preliminary results. Section 3 derives strong-consistency of the estimator introduced in Bosq (2000), in the trace norm. Section 4 formulates a strongly-consistent diagonal componentwise estimator of the autocorrelation operator. Proofs of the results are given in the Supplementary Material.

## 2. Preliminaries

Let $H$ be a real separable Hilbert space, and let $X = \{X_n,\ n \in \mathbb{Z}\}$ be a zero-mean ARH(1) process on the probability space $(\Omega, \mathcal{A}, P)$, satisfying:

$$X_n = \rho\,(X_{n-1}) + \varepsilon_n,\ n \in \mathbb{Z}, \tag{1}$$

where $\rho \in \mathcal{L}(H)$, with $\mathcal{L}(H)$ being the space of bounded linear operators, with the uniform norm $\|\mathcal{A}\|_{\mathcal{L}(H)} = \sup_{f\in H;\ \|f\|_H\leq 1} \mathcal{A}(f)$, for every $\mathcal{A} \in \mathcal{L}(H)$. In our case, $\rho \in \mathcal{L}(H)$ satisfies $\|\rho^k\|_{\mathcal{L}(H)} < 1$, for $k \geq k_0$, and for some $k_0$, where $\rho^k$ denotes the $k$th power of $\rho$, i.e., the composition operator $\rho \underset{k}{\ldots} \rho$. The $H$-valued innovation process $\varepsilon = \{\varepsilon_n,\ n \in \mathbb{Z}\}$ is assumed to be a strong white noise, and to be uncorrelated with the random initial condition. $X$ then admits the MAH($\infty$) representation $X_n = \sum_{k=0}^{\infty} \rho^k\,(\varepsilon_{n-k})$ , for $n \in \mathbb{Z}$, providing the unique stationary solution to Eq. (1) (see Bosq (2000)).

The trace autocovariance operator of $X$ is given by $C_X = \mathrm{E}[X_n \otimes X_n] = \mathrm{E}[X_0 \otimes X_0]$, for $n \in \mathbb{Z}$, and its empirical version $\mathcal{C}_n$ is defined as

$$\mathcal{C}_n = \frac{1}{n}\sum_{i=0}^{n-1} X_i \otimes X_i, \quad n \geq 2, \tag{2}$$

where, for $f \in H$, and $i, j \in \mathbb{N}$, the random operator $X_i \otimes X_j$ is given by $\left(X_i \otimes X_j\right)(f) = \langle X_i, f\rangle_H X_j$. In the following, $\left\{C_j,\ j \geq 1\right\}$ and $\{\phi_j,\ j \geq 1\}$ denote the respective sequence of eigenvalues and eigenvectors of the autocovariance operator $C_X$, satisfying $C_X(\phi_j) = C_j\phi_j$, for $j \geq 1$. Also, by $\{C_{n,j},\ j \geq 1\}$ and $\{\phi_{n,j},\ j \geq 1\}$ we respectively denote the empirical eigenvalues and eigenvectors of $\mathcal{C}_n$ (see Bosq (2000), pp. 102–103),

$$\mathcal{C}_n\phi_{n,j} = C_{n,j}\phi_{n,j},\ j \geq 1, \quad C_{n,1} \geq \cdots \geq C_{n,n} \geq 0 = C_{n,n+1} = C_{n,n+2} \ldots \tag{3}$$

Consider now the nuclear cross-covariance operator $D_X = \mathrm{E}[X_i \otimes X_{i+1}] = \mathrm{E}[X_0 \otimes X_1]$, $i \in \mathbb{Z}$, and its empirical version $\mathcal{D}_n = \frac{1}{n-1}\sum_{i=0}^{n-2} X_i \otimes X_{i+1}$, $n \geq 2$.

The following assumption will appear in the subsequent development.

**Assumption A1.** The random initial condition $X_0$ of $X$ in (1) satisfies $\|X_0\|_H < M$, *a.s.*, for some $M$. Here, a.s. denotes almost surely.

**Theorem 1** (*See Theorem 4.1 on pp. 98–99, Corollary 4.1 on pp. 100–101 and Theorem 4.8 on pp. 116–117, in Bosq (2000)*). *If* $\mathrm{E}\left[\|X_0\|_H^4\right] < \infty$, *for any* $\beta > \frac{1}{2}$, *as* $n \to \infty$,

$$\frac{n^{1/4}}{(\ln(n))^{\beta}}\|\mathcal{C}_n - C_X\|_{\mathcal{S}(H)} \to^{a.s.} 0, \quad \frac{n^{1/4}}{(\ln(n))^{\beta}}\|\mathcal{D}_n - D_X\|_{\mathcal{S}(H)} \to^{a.s.} 0, \tag{4}$$

*where* $\to^{a.s.}$ *means almost surely convergence. Under Assumption A1,*

$$\|\mathcal{C}_n - C_X\|_{\mathcal{S}(H)} = \mathcal{O}\left(\left(\frac{\ln(n)}{n}\right)^{1/2}\right) \; a.s.,$$

$$\|\mathcal{D}_n - D_X\|_{\mathcal{S}(H)} = \mathcal{O}\left(\left(\frac{\ln(n)}{n}\right)^{1/2}\right) \; a.s., \tag{5}$$

*where* $\|\cdot\|_{\mathcal{S}(H)}$ *is the Hilbert–Schmidt operator norm.*

Let $k_n$ be a truncation parameter such that $\lim_{n\to\infty} k_n = \infty$, $\frac{k_n}{n} < 1$, and

$$\Lambda_{k_n} = \sup_{1\leq j\leq k_n} (C_j - C_{j+1})^{-1}. \tag{6}$$

## 3. Strong-consistency in the trace operator norm

This section derives the strong-consistency of the componentwise estimator $\widetilde{\rho}_{k_n}$ (see Eq. (9)), in the trace norm, which also implies its strong-consistency in the Hilbert–Schmidt operator norm. As it is well-known, for a trace operator $\mathcal{K}$ on $H$, its trace norm $\|\mathcal{K}\|_1$ is finite, and, for an orthonormal basis $\{\varphi_n,\ n \geq 1\}$ of $H$, such a norm is given by

$$\|\mathcal{K}\|_1 = \sum_{n=1}^{\infty}\left\langle \sqrt{\mathcal{K}^{\star}\mathcal{K}}(\varphi_n), \varphi_n\right\rangle_H. \tag{7}$$

In Theorem 2, the following lemma will be applied:

**Lemma 1.** *Under Assumption A1, if, as* $n \to \infty$, $k_n\Lambda_{k_n} = o\left(\sqrt{\frac{n}{\ln(n)}}\right)$,

$$\sup_{x\in H,\ \|x\|_H\leq 1}\left\|\rho(x) - \sum_{j=1}^{k_n}\left\langle \rho(x), \phi_{n,j}\right\rangle_H \phi_{n,j}\right\|_H \to_{a.s.} 0, \quad n \to \infty. \tag{8}$$

The proof of this lemma is given in the Supplementary Material.

The following condition is assumed in the remainder of this section:

**Assumption A2.** The empirical eigenvalue $C_{n,k_n} > 0$ *a.s*, where $k_n$ denotes the truncation parameter introduced in the previous section.

Under Assumption A2, from the observations of $X_0, \ldots, X_{n-1}$, consider the componentwise estimator $\widetilde{\rho}_{k_n}$ of $\rho$ (see (8.59) p. 218 in Bosq (2000))

$$\widetilde{\rho}_{k_n}(x) \underset{H}{=} \widetilde{\pi}^{k_n}\mathcal{D}_n[\mathcal{C}_n[\widetilde{\pi}^{k_n}]^{\star}]^{-1}(x) = \widetilde{\pi}^{k_n}\mathcal{D}_n\widetilde{\mathcal{C}}_n^{-1}(x)$$

$$\underset{H}{=} \sum_{j=1}^{k_n}\sum_{p=1}^{k_n}\left\langle \mathcal{D}_n\mathcal{C}_n^{-1}(\phi_{n,j}), \phi_{n,p}\right\rangle_H \phi_{n,p}\left\langle \phi_{n,j}, x\right\rangle_H, \quad \forall x \in H, \tag{9}$$

where $\widetilde{\mathcal{C}}_n^{-1}$ is the inverse of the restriction of $\mathcal{C}_n$ to its principal eigenspace of dimension $k_n$, $\overline{\mathrm{Sp}}^{\|\cdot\|_H}\{\phi_{n,j};\ j = 1, \ldots k_n\} \subseteq H$, which is bounded under Assumption A2, where $\overline{\mathrm{Sp}}^{\|\cdot\|_H}$ denotes the closed span in the norm of $H$. Here, $[\widetilde{\pi}^{k_n}]^{\star}$ denotes the projection operator into $\overline{\mathrm{Sp}}^{\|\cdot\|_H}\{\phi_{n,j};\ j = 1, \ldots k_n\}$, and $\widetilde{\pi}^{k_n}$ is its adjoint or inverse.

**Theorem 2.** *Let $\rho \in \mathcal{L}(H)$ be the autocorrelation operator defined as before. Assume $\Lambda_{k_n}$ in (6) satisfies $\sqrt{k_n}\Lambda_{k_n} = o\left(\frac{n^{1/4}}{(\ln(n))^{\beta}}\right)$ as $n \to \infty$, for $\beta > 1/2$. Then, for $\widetilde{\rho}_{k_n}$ in (9), the following assertions hold:*

***(i)*** *If $\mathrm{E}\left[\|X_0\|_H^4\right] < \infty$, under Assumption A2,*

$$\|\widetilde{\rho}_{k_n} - \widetilde{\pi}^{k_n}\rho[\widetilde{\pi}^{k_n}]^{\star}\|_1 \to^{a.s.} 0, \quad n \to \infty. \tag{10}$$

***(ii)*** *Under Assumptions A1–A2, if $\rho$ is a trace operator, then,*

$$\|\widetilde{\rho}_{k_n} - \rho\|_1 \to^{a.s.} 0, \quad n \to \infty. \tag{11}$$

The proof of this result is given in the Supplementary Material.

The strong consistency in $H$ of the associated ARH(1) plug-in predictor $\widetilde{\rho}_{k_n}(X_{n-1})$ of $X_n$ then follows (see also Bosq (2000) and the Supplementary Material).

## 4. A strongly-consistent diagonal componentwise estimator

In this section, we consider the following assumption:

**Assumption A3.** Assume that $C_X$ is strictly positive, i.e., $C_j > 0$, for every $j \geq 1$, and $D_X$ is a nuclear operator such that $\rho = D_X C_X^{-1}$ is compact.

Under Assumption A3, $\rho$ admits the singular value decomposition(svd)

$$\rho(x) \underset{H}{=} \sum_{j=1}^{\infty} \rho_j \left\langle x, \psi_j\right\rangle_H \widetilde{\psi}_j, \quad \forall x \in H, \tag{12}$$

where, for every $j \geq 1$, $\rho(\psi_j) = \rho_j \widetilde{\psi}_j$, with $\rho_j \in \mathbb{C}$ being the singular value, and $\psi_j$ and $\widetilde{\psi}_j$ the right and left eigenvectors, respectively. Since $D_X$ is a nuclear operator, it admits the svd $D_X(h) \underset{H}{=} \sum_{j=1}^{\infty} d_j \left\langle h, \varphi_j\right\rangle_H \widetilde{\varphi}_j$, $h \in H$, where $\{\varphi_j,\ j \geq 1\}$ and $\{\widetilde{\varphi}_j,\ j \geq 1\}$ are the respective right and left eigenvectors of $D_X$, and $d_j$, $j \geq 1$, are the singular values. $\mathcal{D}_n$ is also nuclear, and $\mathcal{D}_n(h) \underset{H}{=} \sum_{j=1}^{\infty} d_{n,j} \left\langle h, \varphi_{n,j}\right\rangle_H \widetilde{\varphi}_{n,j}$, $h \in H$, with $\{\varphi_{n,j},\ j \geq 1\}$ and $\{\widetilde{\varphi}_{n,j},\ j \geq 1\}$ being the right and left eigenvectors, respectively, and $d_{n,j}$, $j \geq 1$, the singular values. Applying Lemma 4.2, on p. 103, in Bosq (2000),

$$\sup_{j\geq 1} |C_j - C_{n,j}| \leq \|C_X - \mathcal{C}_n\|_{\mathcal{L}(H)} \leq \|C_X - \mathcal{C}_n\|_{\mathcal{S}(H)} \to_{a.s.} 0,\ n \to \infty$$
$$\sup_{j\geq 1} |d_j - d_{n,j}| \leq \|D_X - \mathcal{D}_n\|_{\mathcal{S}(H)} \to_{a.s.} 0,\ n \to \infty. \tag{13}$$

From Theorem 1 (see Eq. (13)), under the conditions assumed in such a theorem, for $n$ sufficiently large, in view of Assumption A3, the composition operator $\mathcal{D}_n\mathcal{C}_n^{-1}$ is compact on $H$, admitting the svd

$$\mathcal{D}_n\mathcal{C}_n^{-1}(h) = \sum_{j=1}^{n} \widehat{\rho}_{n,j}\widetilde{\psi}_{n,j} \left\langle h, \psi_{n,j}\right\rangle_H, \quad \forall h \in H, \tag{14}$$

where $\mathcal{D}_n\mathcal{C}_n^{-1}(\psi_{n,j}) = \widehat{\rho}_{n,j}\widetilde{\psi}_{n,j}$, for $j = 1, \ldots, n$, with $\{\psi_{n,j},\ j \geq 1\}$ and $\{\widetilde{\psi}_{n,j},\ j \geq 1\}$ being the empirical right and left eigenvectors of $\rho$.

**Proposition 1.** *Under conditions in Theorem 2**(ii)**, and Assumption A3,*

$$\|\mathcal{D}_n\mathcal{C}_n^{-1} - D_X C_X^{-1}\|_{\mathcal{L}(H)} \to_{a.s.} 0, \quad n \to \infty. \tag{15}$$

The proof of this proposition directly follows from

$$\begin{aligned}
&\sup_{x\in H:\|x\|_H\leq 1} \|\mathcal{D}_n\mathcal{C}_n^{-1}(x) - D_X C_X^{-1}(x)\|_H \\
&\leq 2\|\mathcal{D}_n\mathcal{C}_n^{-1}\|_{\mathcal{L}(H)} \left[\sum_{j=1}^{k_n} \|\phi'_{n,j} - \phi_{n,j}\|_H + \sum_{j=k_n+1}^{\infty} \left\|\phi'_{n,j}\right\|_H\right] \\
&+\|\widetilde{\rho}_{k_n} - D_X C_X^{-1}\|_{\mathcal{L}(H)} \to_{a.s.} 0, \quad n \to \infty,
\end{aligned} \tag{16}$$

where $\phi'_{n,j} = sgn\langle\phi_j, \phi_{n,j}\rangle_H\phi_j$, with $sgn\langle\phi_j, \phi_{n,j}\rangle_H = \mathbf{1}_{\langle\phi_j,\phi_{n,j}\rangle_H\geq 0} - \mathbf{1}_{\langle\phi_j,\phi_{n,j}\rangle_H< 0}$. Under Assumption A3, Eq. (15) holds, if the conditions assumed in Bosq (2000) for the strong-consistency of $\widetilde{\rho}_{k_n}$ in $\mathcal{L}(H)$ hold. From Proposition 1, and (12) and (14), applying Lemma 4.2, on p. 103 in Bosq (2000),

$$\sup_{j\geq 1} \left|\widehat{\rho}_{n,j} - \rho_j\right| \leq \|\mathcal{D}_n\mathcal{C}_n^{-1} - D_X C_X^{-1}\|_{\mathcal{L}(H)} \to^{a.s.} 0,\ n \to \infty. \tag{17}$$

Let us define the following quantity:

$$\Lambda_{k_n}^{\rho} = \sup_{1\leq j\leq k_n} (|\rho_j|^2 - |\rho_{j+1}|^2)^{-1}, \tag{18}$$

where $k_n$ denotes the truncation parameter introduced in Section 2. We now apply the methodology of the proof of Lemma 4.3, on p. 104, and Corollary 4.3, on p. 107, in Bosq (2000), to obtain the strong-consistency of the empirical right and left eigenvectors, $\{\psi_{n,j},\ j\geq 1\}$ and $\{\widetilde{\psi}_{n,j},\ j\geq 1\}$ of $\rho$, under the following additional assumption:

**Assumption A4.** Consider $\left[\sup_{j\geq 1}|\rho_j| + \sup_{j\geq 1}|\widehat{\rho}_{n,j}|\right] \leq 1.$

**Lemma 2.** *Under Assumptions A3–A4, and the conditions of Theorem 2**(ii)**, if $\Lambda_{k_n}^{\rho}$ in (18) satisfies $\Lambda_{k_n}^{\rho} = o\left(\frac{1}{M_n}\right)$, with $M_n\in\mathbb{R}$ such that $\|\mathcal{D}_n\mathcal{C}_n^{-1} - D_X C_X^{-1}\|_{\mathcal{L}(H)} = \mathcal{O}(M_n)$, a.s., as $n\to\infty$, then,*

$$\sup_{1\leq j\leq k_n}\|\psi_{n,j} - \psi'_{n,j}\|_H \to_{a.s.} 0, \qquad \sup_{1\leq j\leq k_n}\|\widetilde{\psi}_{n,j} - \widetilde{\psi}'_{n,j}\|_H \to_{a.s.} 0, \tag{19}$$

*where, for $j\geq 1,\ n\geq 2,\ \psi'_{n,j} = sgn\left\langle\psi_{n,j},\psi_j\right\rangle_H \psi_j \quad \widetilde{\psi}'_{n,j} = sgn\left\langle\widetilde{\psi}_{n,j},\widetilde{\psi}_j\right\rangle_H \widetilde{\psi}_j$, with $sgn\langle\psi_{n,j},\psi_j\rangle_H = \mathbf{1}_{\langle\psi_{n,j},\psi_j\rangle_H\geq 0} - \mathbf{1}_{\langle\psi_{n,j},\psi_j\rangle_H<0}$ and $sgn\langle\widetilde{\psi}_{n,j},\widetilde{\psi}_j\rangle_H = \mathbf{1}_{\langle\widetilde{\psi}_{n,j},\widetilde{\psi}_j\rangle_H\geq 0} - \mathbf{1}_{\langle\widetilde{\psi}_{n,j},\widetilde{\psi}_j\rangle_H<0}$.*

The proof of this lemma is given in the Supplementary Material.

The following diagonal componentwise estimator $\widehat{\rho}_{k_n}$ of $\rho$ is formulated:

$$\widehat{\rho}_{k_n}(x) = \sum_{j=1}^{k_n}\widehat{\rho}_{n,j}\left\langle x,\psi_{n,j}\right\rangle_H \widetilde{\psi}_{n,j}, \quad \forall x\in H. \tag{20}$$

The next result derives the strong-consistency of $\widehat{\rho}_{k_n}$.

**Theorem 3.** *Under the conditions of Lemma 2, if $k_n\Lambda_{k_n}^{\rho} = o\left(\frac{1}{M_n}\right)$, with $M_n\in\mathbb{R}$ such that $\|\mathcal{D}_n\mathcal{C}_n^{-1} - D_X C_X^{-1}\|_{\mathcal{L}(H)} = \mathcal{O}(M_n)$, a.s., as $n\to\infty$, then, $\|\widehat{\rho}_{k_n} - \rho\|_{\mathcal{L}(H)} \to_{a.s.} 0$, as $n\to\infty$.*

The proof of this result is given in the Supplementary Material.

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## Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spl.2018.09.004.

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