Chapter 4
A diagonal componentwise approach for ARB(1) prediction

Javier Álvarez-Liébana and M. Dolores Ruiz-Medina

Abstract This paper extends to the Banach-valued framework previous strong-consistency results derived, in the context of diagonal componentwise estimation of the autocorrelation operator of autoregressive Hilbertian processes, and the associated plug-in prediction. The Banach space B considered here is $B = \mathscr{C}([0,1])$, the space of continuous functions on $[0,1]$ with the supremum norm.

4.1 Introduction.

Functional time series theory emerges as a powerful tool in the statistical analysis of high dimensional data correlated in time. In the Hilbert-valued context, we refer to the reader to the book by Bosq [6], and the papers by Mas [14]; Guillas [9]; Kargin and Onatski [10], among others; and, more recently, to the papers [1, 2, 8, 18, 19]. In particular, in the autoregressive Hilbertian framework (ARH(1) framework), Álvarez-Liébana, Bosq and Ruiz-Medina [2] prove weak-consistency results, in the space of Hilbert-Schmidt operators, for a diagonal componentwise estimator of the autocorrelation operator, and its associated plug-in predictor (in the underlying Hilbert space H). For the same type of diagonal componentwise estimator of the autocorrelation operator, and its associated ARH(1) plug-in predictor, strong-consistency results in the space of bounded linear operators, and in the underlying Hilbert space H, are respectively obtained in Álvarez-Liébana and Ruiz-Medina [3].

Estimation and prediction in the context of Banach-valued autoregressive processes of order one (ARB(1) processes) have also been widely developed, when $B = \mathscr{C}([0,1])$. In the estimation of ARC(1) processes, and its associated ARC(1) plug-in prediction, strong-consistency results are derived in Pumo [16, 17] and Bosq [6], under suitable regularity conditions. In Labbas and Mourid [13], Kuelb's Lemma

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plays a key role to extend the results presented in [6, 16, 17]. The formulation of the previous results to the case of weakly dependent innovation process is considered in Dehling and Sharipov [7]. The estimation of ARB(p) processes, with p greater than one, is addressed, for example, in Mourid [15].

Here, we adopt the approach presented in [6, 13] in the framework of ARC(1) processes. Specifically, the extension, from a Banach space to a separable Hilbert space, of the autocorrelation operator of an ARB(1) process (in particular, ARC(1) process) is achieved. The diagonal formulation of the componentwise estimator of the extended autocorrelation operator is considered in this paper, as given in Álvarez-Liébana, Bosq and Ruiz-Medina [2]. A key feature of the diagonal design is the important dimension reduction achieved, with a better or equivalent performance in relation to the approaches presented in [4, 5, 6, 9], as shown in the simulation study undertaken in Álvarez-Liébana, Bosq and Ruiz-Medina [2]. The outline of this work is as follows. Preliminaries section describes the ARB(1) framework. A diagonal component-wise estimator of the extended autocorrelation operator is introduced in Section 4.2. Strong-consistency results, in the space of bounded linear operators on a Hilbert space, are detailed in Section 4.3. The formulation of such results in the Banach-valued context, considering the case of estimation of ARC(1) processes, in the norm of bounded linear operators over $\mathscr{C}([0,1])$, is analyzed in Section 4.4. Final comments and open research lines are given in Section 4.5.

4.2 Preliminaries: ARB(1) general framework and estimator of autocorrelation operator.

Let us consider $(B, \|\cdot\|_B)$ as a real separable Banach space, and let $X = \{X_n,\ n \in \mathbb{Z}\}$ be a zero-mean ARB(1) process, associated with $(\mu, \varepsilon = \{\varepsilon_n,\ n \in \mathbb{Z}\}, \rho)$, satisfying the following equation (see [6], p. 148):

$$X_n = \rho\left(X_{n-1}\right) + \varepsilon_n, \quad n \in \mathbb{Z}, \tag{4.1}$$

where ρ is the autocorrelation operator of X, belonging to the space $\mathscr{L}(B)$ of bounded linear operators on the Banach space B, such that $\|\rho\|_{\mathscr{L}(B)} < 1$. The innovation process ε is assumed to be a strong white noise, and to be also uncorrelated with the random initial condition, with $\sigma_\varepsilon^2 = \mathrm{E}\left[\|\varepsilon_n\|_B^2\right] < \infty$, for all $n \in \mathbb{Z}$. Note that $E\left[\|X_n\|_B^2\right] < \infty$, for all $n \in \mathbb{Z}$, under the above conditions assumed in the introduction of equation (4.1) (see [6], Theorem 6.1).

The autocovariance and cross-covariance operators of X, denoted as C_X and D_X, respectively, are defined as follows, for all $f, g \in B^*$:

$$C_X(f)(g) = \mathrm{E}\left[f(X_n)\, g(X_n)\right], \quad D_X(f)(g) = \mathrm{E}\left[f(X_n)\, g(X_{n+1})\right], \quad n \in \mathbb{Z}, \tag{4.2}$$

where B^* denotes the dual space of B.

Assumption A1. The operator C_X, defined in (4.2), is a nuclear operator given by

$$C_X(f) = \sum_{j=1}^{\infty} f(x_j)\, x_j, \quad \sum_{j=1}^{\infty} \|x_j\|_B^2 < \infty, \quad \forall f \in B^*, \tag{4.3}$$